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Plongements polyédriques du tore carré plat

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Thèse de doctorat

Université Claude Bernard Lyon 1

THESE de DOCTORAT DE L'UNIVERSITE DE LYON

opérée au sein de
l'Université Claude Bernard Lyon 1

Ecole Doctorale InfoMaths, ED 512

Spécialité : Mathématiques
N. d'ordre xx–xxxx

Soutenue publiquement le xx mois xxxx par :

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Chapter 1

Introduction

The goal of this work is to study PL isometric embeddings of the square flat torus in Euclidean space $\mathbb{E}^3 = (\mathbb{R}^3, \langle\, , \,\rangle)$. In 2012, a C^1 isometric embedding of the square flat torus in $\mathbb{E}^3$ was explicitly constructed in [**?**, **?**]. Recall that the square flat torus is the quotient $\mathbb{T}^2 = \mathbb{E}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$ where (e_1, e_2) is an orthonormal basis of $\mathbb{E}^2$. An embedding $f : \mathbb{T}^2 \to \mathbb{E}^3$ is isometric if the pullback by f of the metric induced by $\mathbb{E}^3$ on $f(\mathbb{T}^2)$ coincides with the Euclidean metric on $\mathbb{T}^2$. Here, we are interested in piecewise linear (PL) embeddings instead of C^1 embeddings.

A **PL embedding** is a linear embedding into $\mathbb{E}^3$ of a simplicial complex C that triangulates $\mathbb{T}^2$. Let $h : C \to \mathbb{T}^2$ be such a triangulation. A **PL isometric embedding** of $\mathbb{T}^2$ is a linear embedding $g : C \to \mathbb{E}^3$ that induces an isometry between C and $f(C)$ where C is endowed with the pullback by h of the Euclidean metric on $\mathbb{T}^2$ and $f(C)$ is equipped with the metric induced by $\mathbb{E}^3$. Observe that h and g define an isometric map $f : \mathbb{T}^2 \to \mathbb{E}^3$ according to the following diagram.

$$\begin{array}{ccc} C & \xrightarrow{h} & \mathbb{T}^2 \\ & \searrow^{g} & \downarrow f \\ & & \mathbb{E}^3 \end{array}$$

We finally recall that a linear embedding of a simplicial complex sends each edge to a linear segment in $\mathbb{E}^3$. In 1996 Burago and Zalgaller [**?**] proved that any connected compact polyhedral surface admits an PL isometric embedding into $\mathbb{E}^3$. The year after Zalgaller [**?**] constructed explicit PL isometric embeddings of long cylinders and long flat tori, but up to now no explicit PL embedding of the *square* flat torus was known. In the first part of this thesis we provide such an embedding, see Figure 1.1. More precisely we prove the following

Theorem 1.0.1. *There exists a PL isometric embedding of the square flat torus with at most 48 vertices.*

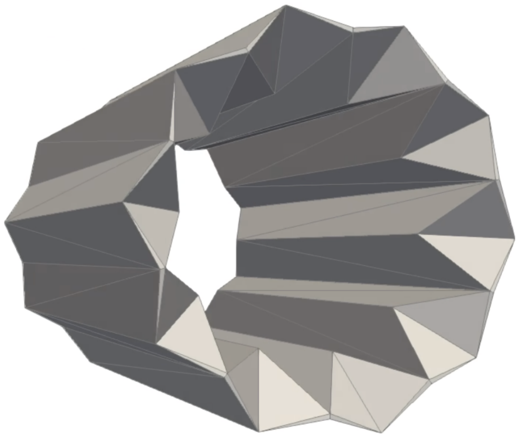

Figure 1.1: A PL isometric embedding of the square flat torus.

The idea behind the construction is to use corrugations and was inspired by the convex integration theory. This result raises the question of the minimal number of vertices needed a PL isometric embedding of the square flat torus. In other words, we look for the smallest triangulation of the torus that admits a linear embedding isometric to $\mathbb{T}^2$. The minimal triangulation is known as the Moebius torus. Its one-skeleton (vertex-edge graph) is the complete graph on 7 vertices. It admits a linear embedding, the so-called Császár torus [Csá49] which is not isometric to any flat torus. See Figure 1.2 We conjecture that the Moebius torus has no linear embedding isometric to $\mathbb{T}^2$.

In order to understand this conjecture, we first note that in the above diagram, the map h induces a geodesic triangulation $\mathcal{T}$ on $\mathbb{T}^2$ such that every edge is a distance minimizing geodesic i.e. a geodesic segment. Since $\mathbb{E}^2$ acts isometrically by translations on the set of geodesic triangulations of $\mathbb{T}^2$ we consider the space $\mathrm{GE}(C, \mathbb{T}^2)$ of geodesic triangulations of $\mathbb{T}^2$ isomorphic to C modulo this action. In particular $\mathcal{T}$ (modulo translations) is an element of $\mathrm{GE}(C, \mathbb{T}^2)$. When $C = M$ we are able to give a complete description of the space of configurations $\mathrm{GE}(M, \mathbb{T}^2)$.

Theorem 1.0.2. *The configuration space $GE(M, \mathbb{T}^2)$ is the disjoint union of 12 products of simplices*

$$GE(M, \mathbb{T}^2) = \bigcup_{i=1}^{12} \Delta_x^i \times \Delta_y^i,$$

where Δ_x^i, Δ_y^i are 6-dimensional simplices.

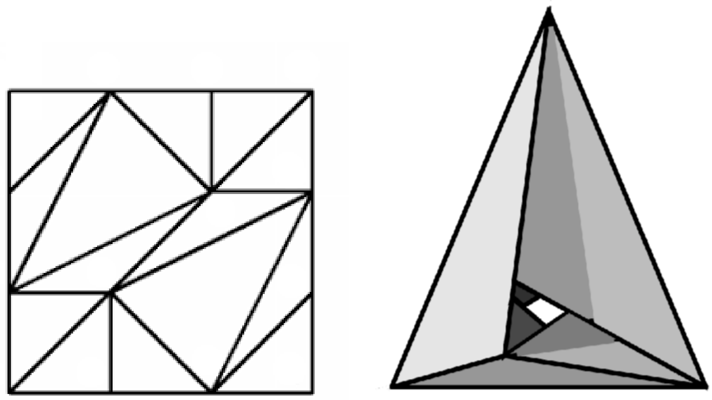

Figure 1.2: Left, the Moebius triangulation of the torus. Right, the Császár torus.

Let $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. We denote by $\mathrm{L}_{\mathcal{T}}(\mathbb{E}^3)$ the set of linear isometric embeddings of $\mathcal{T}$ in $\mathbb{E}^3$ and by $\mathrm{L}_{\mathbb{T}^2}(M, \mathbb{E}^3)$ the union of $\mathrm{L}_{\mathcal{T}}(\mathbb{E}^3)$ over all $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. Proving that the Moebius torus has no linear embedding isometric to $\mathbb{T}^2$ is thus equivalent to show that the space $\mathrm{L}_{\mathbb{T}^2}(M, \mathbb{E}^3)$ is empty, i.e. that $\mathrm{L}_{\mathcal{T}}(\mathbb{E}^3)$ is empty for every $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. We were able to prove this result for a relatively large subspace of $\mathrm{GE}(M, \mathbb{T}^2)$.

Theorem 1.0.3. *For each $i = 1, \ldots, 12$, there exists a section $f_i : \Delta_x^i \to \Delta_x^i \times \Delta_y^i$ whose image $S_i = f_i(\Delta_x^i)$ has a 12-dimensional neighborhood $N(S_i)$ such that for every element $\mathcal{T} \in N(S_i)$ we have $L_{\mathcal{T}}(\mathbb{E}^3) = \emptyset$.*

This result is already non trivial and relies on a tight estimation of some geometric quantities associated to sub-configurations of each $\mathcal{T} \in S_i$.

Chapter 2

PL-embedding of the Flat Torus into $\mathbb{E}^3$

We present a 2-parameter family of explicit PL-embeddings of the flat square torus $\mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}^2$ into $\mathbb{E}^3$. One of them only involves 48 vertices.

2.1 Introduction

In [BZ95], Burago and Zalgaller proved that any connected compact polyhedral surface admits an isometric piecewise linear (PL) immersion into $\mathbb{E}^3$. Recall that a polyhedral surface is 2-dimensional manifold endowed with a polyhedral metric i. e. a metric such that every point has a neighborhood isometric to the neighborhood of the vertex of a cone in $\mathbb{E}^3$. Their approach relies on the Nash-Kuiper C^1-embedding Theorem ([Nas54] and [Kui54]) and their construction is not explicit for this reason (see [Sau12] for a discussion). In addition to an initial PL-approximation of an almost C^1 isometric embedding, the construction of Burago-Zalgaller involves several subdivision steps so that the resulting triangulation is very large. Finding an explicit triangulation with few vertices appears a real challenge. Zalgaller investigated the question of how to construct explicit PL-embeddings of cylinders or flat tori and found a solution for long cylinders and long tori [Zal00]. Recall that a flat torus is the quotient of the two-dimensional Euclidean plane by a lattice. It is called rectangular when the lattice itself is rectangular. The above construction of Zalgaller restricted to rectangular tori requires that the width is at least twice of its height. In this article, we provide the first explicit PL-embeddings of short rectangle tori including the square torus.

Theorem 2.1.1. *There exists a PL isometric embedding of any rectangular torus with at most 48 vertices.*

Our PL-embeddings are inspired from the corrugated C^1 isometric embed-

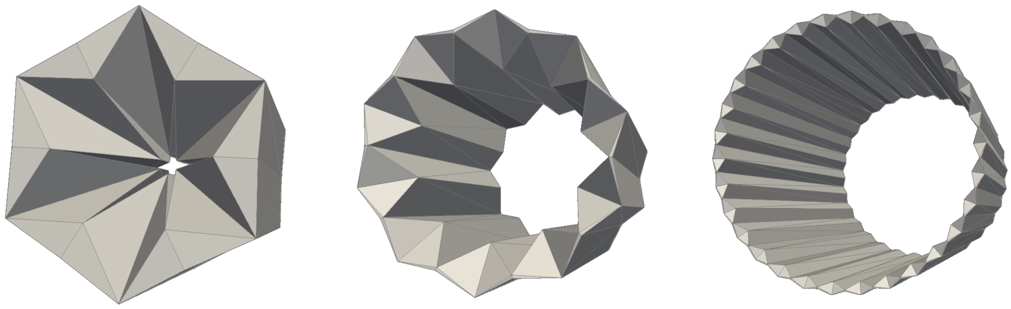

Figure 2.1: PL isometric embeddings of the square flat torus with 6, 10 and 30 corrugations ($x = 0.1$, see below)

dings of the flat torus generated by the Convex Integration Theory and constructed in [BJLT13] and [BJLT12]. Essentially, we construct PL corrugations along one side of the rectangle to introduce flexibility and to allow the identifications between the opposite sides. We show that six corrugations are enough to obtain a PL-isometric embedding. The corresponding number of vertices is 48. If the isometric constraint is released, it is known that the torus admits a PL-embedding with 7 vertices and that this number can not be reduced [Csá49] and [BE91]. The question of the minimum number of vertices of a PL-isometric embedding of a flat torus is quite natural but probably very difficult.

2.2 Triangulations of the square torus

In this section we describe a family of triangulations $\mathcal{T}(x,n)$ of the square torus depending on two parameters, $n \in \mathbb{N}_{\geq 6}$ and $x \in]0, 0.15[$.

General description of $\mathcal{T}(x,n)$.– Let (ϵ_1, ϵ_2) be an orthonormal basis of $\mathbb{E}^2$ and let $\mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ be the square torus. We consider $\pi : \mathbb{E}^2 \to \mathbb{T}^2$ the covering map and $\mathcal{D} := [0,1[^2 \subset \mathbb{E}^2$ a fundamental domain. We identify the triangulation $\mathcal{T}(x,n)$ with its trace on $\mathcal{D}$, namely with $\pi^{-1}(\mathcal{T}(x,n)) \cap \mathcal{D}$. The triangulation is built from a pattern which consists of 8 triangles located on vertical ribbon of width $\frac{1}{2n}$. This pattern is then reflected and translated to create the whole triangulation (see figure 2.2).
Each ribbon will be mapped into $\mathbb{R}^3$ to generate half of a PL corrugation (see Figure 4). The rotational symmetry of the embedding combined with the isometric constraints greatly compel the geometry of the trapezoids $a_ib_id_ic_i$. They are, for this reason, the basic pieces of the triangulation and we now

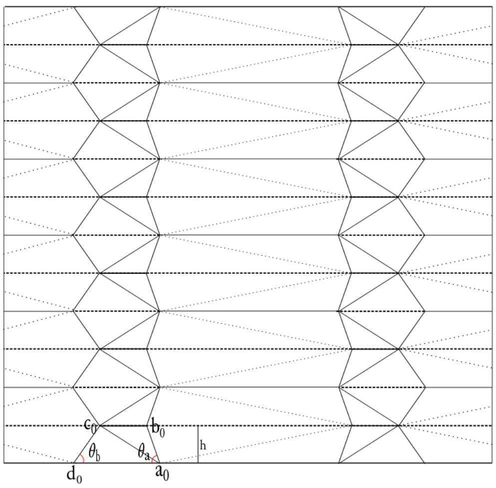

Figure 2.2: The triangulation $\mathcal{T}(x,n)$ (the initial ribbon is in grey).

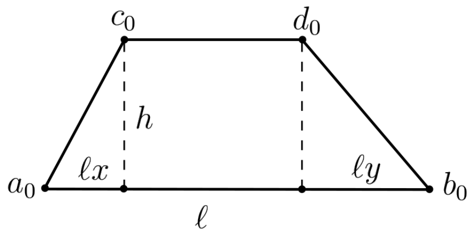

Figure 2.3: Basic trapezoid (view with a quarter turn with respect to Figure 2.2).

describe them.

The basic piece of $\mathcal{T}(x,n)$.– We consider the trapezoid $a_0b_0c_0d_0$ and we denote by ℓ its larger basis, by h its height and by θ_a and θ_b its angles at a_0 and b_0. We assume and $0 < \theta_a \leq \frac{\pi}{2}$ and $0 < \theta_b < \frac{\pi}{2}$. See figure 2.3.
We now give a choice of the quantities h, ℓ, θ_a and θ_b that will fulfill the rotational and isometric constraints. Let $n \in \mathbb{N}_{\geq 6}$ and $x \in]0, 0.15[$, we put

(A1) $h = \frac{1}{2n}$

(A2) $\ell = \Phi(x,h)$ where

$$\Phi(x,h) = \frac{h\delta(x)\sin(2h\pi)}{\sqrt{(\sigma(x) - (x+0.1)\cos(2h\pi)\delta(x))^2 + x^2\sin^2(2h\pi)\delta^2(x)}}$$

with $\delta(x) = \sqrt{1-(2x-0.9)^2}$ and $\sigma(x) = -2x^2 + 1.9x + 0.1$.

(A3) $y = 0.9 - x$

We then locate the trapezoid at positions:

$$a_0 = \left(0, \frac{1}{4}(1 + 2\ell(0.1 + 2x))\right), \quad b_0 = \left(0, \frac{1}{4}(1 + 2\ell(-1.9 + 2x))\right),$$

$$c_0 = \left(h, \frac{1}{4}(1 + 0.2\ell)\right), \quad d_0 = \left(h, \frac{1}{4}(1 - 0.2\ell)\right)$$

The triangulation $\mathcal{T}(x, n)$**.–** We build another trapezoid $a'_0 b'_0 c'_0 d'_0$ by reflecting $a_0 b_0 c_0 d_0$ through the line $y = \frac{1}{2}$, we have

$$a'_0 = \left(0, \frac{1}{4}(3 - 2\ell(0.1 + 2x))\right), \quad b'_0 = \left(0, \frac{1}{4}(3 - 2\ell(-1.9 + 2x))\right),$$

$$c'_0 = \left(h, \frac{1}{4}(3 - 0.2\ell)\right), \quad d'_0 = \left(h, \frac{1}{4}(3 + 0.2\ell)\right)$$

Let s be the reflection through the vertical line of abscissa h. We set $a_1 = s(a_0)$, $b_1 = s(b_0), a'_1 = s(a'_0)$ and $b'_1 = s(b'_0)$ and we add the obvious edges to define the two new trapezoids $a_1 b_1 c_0 d_0$ and $a'_1 b'_1 c'_0 d'_0$. Let τ be the translation of vector $2h\epsilon_1$. For every $i \in \{1, ..., n-1\}$ we set $a_i := \tau^i(a_0)$, ..., $d'_i = \tau^i(d'_0)$ and define the trapezoids $a_i b_i c_i d_i$, $a'_i b'_i c'_i d'_i$, $a_{i+1} b_{i+1} c_i d_i$ and $a'_{i+1} b'_{i+1} c'_i d'_i$. See Figure 2.2. We complete the triangulation $\mathcal{T}(x, n)$ by adding for every $i \in \{0, ..., n-1\}$ the Euclidean segments $a_i d_i$, $a'_i d'_i$, $a_i a'_i$, $c_i c'_i$, $a_i c'_i$, $a_{i+1} c'_i$, and four families of edges $b_i b'_i$, $d_i d'_i$, $b_i d'_i$ and $b_{i+1} d'_i$ whose traces on $\mathcal{D}$ are broken into two pieces. Namely the trace of the edge $b_i b'_i$ is $\pi^{-1}(\pi([b_i, b'_i - \epsilon_2])) \cap \mathcal{D}$ and respectively $\pi^{-1}(\pi([d_i, d'_i - \epsilon_2])) \cap \mathcal{D}$ for $d_i d'_i$, $\pi^{-1}(\pi([b_i, d'_i - \epsilon_2])) \cap \mathcal{D}$ for $b_i d'_i$ and $\pi^{-1}(\pi([b_{i+1}, d'_i - \epsilon_2])) \cap \mathcal{D}$ for $b_{i+1} d'_i$.

For a latter use we will need the following lemma.

Lemma 2.2.1. *For every* $n \in \mathbb{N}_{\geq 6}$ *and* $x \in \,]0, 0.15[$ *we have*

$$\ell^2 < \frac{h^2}{\delta^2(x)\sin^2(2\pi h) + x^2}.$$

Proof. After substituting ℓ by its expression $(A2)$, the inequality of the lemma reads:

$$\frac{h^2 \sin^2(2\pi h)\delta^2(x)}{(\sigma(x) - (0.1 + x)\delta(x)\cos(2\pi h))^2 + x^2\delta^2(x)\sin^2(2\pi h)} < \frac{h^2}{\delta^2(x)\sin^2(2\pi h) + x^2}.$$

This last inequality is easily seen to be equivalent to the following one:

$$f_n(x) < \sigma(x) \text{ where } f_n(x) = \delta^2(x)\sin^2\left(\frac{\pi}{n}\right) + \delta(x)\cos\left(\frac{\pi}{n}\right)(0.1 + x).$$

Let $\mathcal{I}_n = \{t \in \,]0, 1] \mid f_n(t) < \sigma(t)\}$. Using that the sequence $(f_n(t))_{n \in \mathbb{N}^*}$ is decreasing, it can be checked that $\mathcal{I}_n = \emptyset$ if $n \leq 5$ and that

$$]0, 0.15[\subset \mathcal{I}_6 \subset ... \subset \mathcal{I}_n.$$

This proves the lemma. □

2.3 PL embeddings of the square torus

In this section we describe for each $n \in \mathbb{N}_{n\geq 6}$ and $x \in]0, 0.15[$ a linear embedding of the triangulation $\mathcal{T}(x,n)$ into $\mathbb{E}^3$. We denote by O the origin of $\mathbb{E}^3$ and we introduce the three following points of $\mathbb{E}^3$:

$$\begin{aligned} \Omega_A &= \left(0, 0, \frac{1}{4}(1 - 2\ell(0.1 + 2x))\right) \\ \Omega_B &= \left(0, 0, \frac{1}{4}(1 - 2\ell(1.9 - 2x))\right) \\ \Omega_* &= \left(0, 0, \frac{1}{4}(1 - 0.2\ell)\right) \end{aligned} \tag{2.1}$$

We define a PL map $F : \mathbb{T}^2 \to \mathbb{E}^3$ by its image on every vertex of $\mathcal{T}(x,n)$:

$$\begin{aligned} F(a_i) &= \Omega_A + r_4 \left(\cos\frac{(2i+1)\pi}{n}, -\sin\frac{(2i+1)\pi}{n}, 0\right) \\ F(b_i) &= \Omega_B + r_1 \left(\cos\frac{(2i+1)\pi}{n}, -\sin\frac{(2i+1)\pi}{n}, 0\right) \\ F(c_i) &= \Omega_* + r_3 \left(\cos\frac{2i\pi}{n}, -\sin\frac{2i\pi}{n}, 0\right) \\ F(d_i) &= \Omega_* + r_2 \left(\cos\frac{2i\pi}{n}, -\sin\frac{2i\pi}{n}, 0\right) \end{aligned} \tag{2.2}$$

and similarly,

$$\begin{aligned} F(a_i') &= -\Omega_A + r_4 \left(\cos\frac{(2i+1)\pi}{n}, -\sin\frac{(2i+1)\pi}{n}, 0\right) \\ F(b_i') &= -\Omega_B + r_1 \left(\cos\frac{(2i+1)\pi}{n}, -\sin\frac{(2i+1)\pi}{n}, 0\right) \\ F(c_i') &= -\Omega_* + r_3 \left(\cos\frac{2i\pi}{n}, -\sin\frac{2i\pi}{n}, 0\right) \\ F(d_i') &= -\Omega_* + r_2 \left(\cos\frac{2i\pi}{n}, -\sin\frac{2i\pi}{n}, 0\right) \end{aligned}$$

for all $i \in \{0, \ldots, n-1\}$ and where r_1, r_2, r_3 and r_4 are given by:

$$\begin{aligned} r_4 &= \frac{\sqrt{h^2 - \ell^2 x^2}}{\sin(2\pi h)} \\ r_3 &= r_4 \cos(2\pi h) - \ell x \\ r_2 &= r_3 - c_0 d_0 \\ r_1 &= r_4 - \ell\delta(x). \end{aligned} \tag{2.3}$$

Note that the points $F(a_i)$, $i \in \{0, ..., n\}$, lie in a circle of radius r_4 and of center Ω_A. Similarly, the points $F(b_i)$, $i \in \{0, ..., n\}$, lie in a circle of radius r_1 and of center Ω_B, and so on. See Figure 2.4.

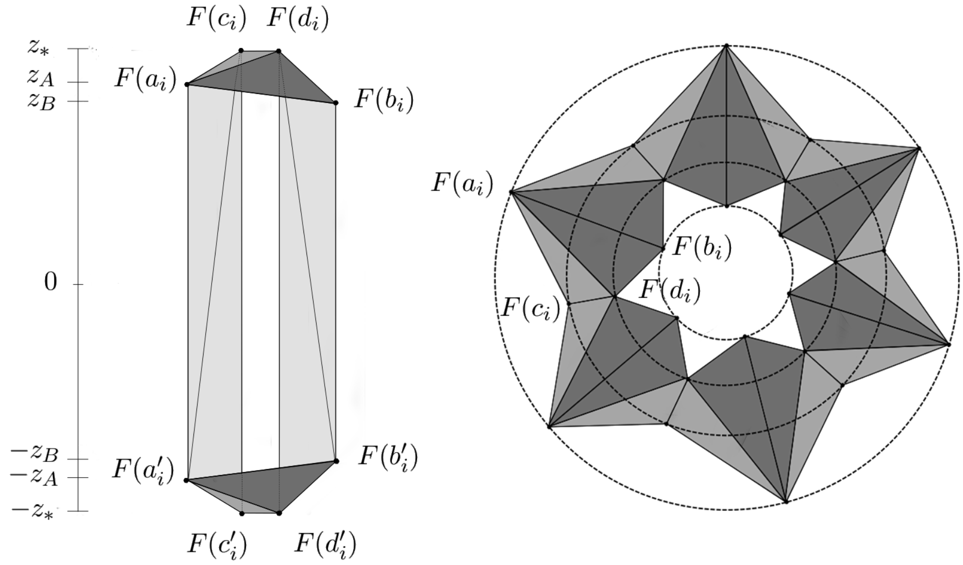

Figure 2.4: Left, view of a slice of $F(\mathbb{T}^2)$, on the vertical axis z_A, z_B and z_* denote the vertical coordinates of Ω_A, Ω_B and Ω_*. Right, view from above of $F(\mathbb{T}^2)$, the circles have radii $r_1 < r_2 < r_3 < r_4$.

Observe that in this figure we have assumed that $0 < r_1 < r_2 < r_3 < r_4$. The fact that these inequalities hold is ensured by the following lemma:

Lemma 2.3.1. *We have:* $0 < r_1 < r_2 < r_3 < r_4$.

Proof. We first note that

$$r_4 - r_3 = r_4 \left(1 - \cos(2\pi h)\right) + \ell x > 0$$

whence $r_4 > r_3$. We also have $r_3 > r_2$ since $r_2 = r_3 - c_0 d_0$. The proof that $r_1 < r_2$ is less straightforward. We have

$$r_1 < r_2 \iff r_4 - \ell\delta(x) < r_3 - c_0 d_0 \iff r_4 - r_3 < \ell\delta(x) - c_0 d_0$$

Since $r_3 = r_4 \cos(2\pi h) - \ell x$ and $c_0 d_0 = 0.1\ell$ the last inequality reduces to

$$r_4(1 - \cos(2\pi h)) < \ell(\delta(x) - 0.1 - x).$$

Replacing r_4 with its value in the above inequality, we obtain

$$\frac{\sqrt{h^2 - \ell^2 x^2}\,(1 - \cos(2\pi h))}{\sin(2\pi h)} < \ell\left(\delta(x) - 0.1 - x\right)$$

Observe that both sides are positive, therefore this inequality is equivalent to

$$(h^2 - \ell^2 x^2)\,(1-\cos(2\pi h))^2 < \ell^2\,(\delta(x) - 0.1 - x)^2 \sin^2(2\pi h)$$

i. e.

$$h^2\,(1-\cos(2\pi h))^2 < \ell^2 \left((\delta(x) - 0.1 - x)^2 \sin^2(2\pi h) + x^2\,(1-\cos(2\pi h))^2\right).$$

By $(A2)$ we obtain

$$(1-\cos(2\pi h))^2 < \frac{\sin^2(2\pi h)\delta^2(x)\left((\delta(x) - 0.1 - x)^2 \sin^2(2\pi h) + x^2\,(1-\cos(2\pi h))^2\right)}{(\sigma(x) - (0.1+x)\delta(x)\cos(2\pi h))^2 + x^2 \sin^2(2\pi h)\delta^2(x)}$$

or equivalently

$$(1-\cos(2\pi h))^2\,(\sigma(x) - (0.1+x)\delta(x)\cos(2\pi h))^2 < \sin^4(2\pi h)\delta^2(x)\,(\delta(x) - 0.1 - x)^2$$

Taking the square root, we obtain

$$(1-\cos(2\pi h))\,(\sigma(x) - (0.1+x)\delta(x)\cos(2\pi h)) < \sin^2(2\pi h)\delta(x)\,(\delta(x) - 0.1 - x)\,.$$

We thus have to prove that

$$\frac{\sigma(x) - (0.1+x)\delta(x)\cos(2\pi h)}{\delta(x)(\delta(x) - 0.1 - x)} < \frac{\sin^2(2\pi h)}{1-\cos(2\pi h)}.$$

To do so, we remark that the right term is greater than 1 and we show that the left term is lower than 1. This will be achieved if

$$g_n(x) := \sigma(x) - (0.1+x)\delta(x)\cos(2\pi h) - \delta(x)(\delta(x) - 0.1 - x)$$

is negative. It is straightforward to see that

$$g_n(x) = 2x^2 - 1.7x - 0.09 + \delta(x)(0.1+x)(1-\cos(2\pi h)).$$

Since $\delta(x) < 1$ for $x \in]0, 0.15[$, we have

$$g_n(x) \le 2x^2 - 1.7x - 0.09 + (0.1+x)\left(1 - \cos\left(\frac{\pi}{6}\right)\right) =: g(x)$$

It is then easily seen that $g(x) < 0$ for every $x \in]0, 0.15[$. It follows that $g_n(x) < 0$ thus proving $r_1 < r_2$.

It remains to prove that $r_1 > 0$. Since $r_1 = r_4 - \ell\delta(x)$, it is enough to show that $\ell^2\delta^2(x) < r_4^2$ i. e.

$$\ell^2\delta^2(x) < \frac{h^2 - x^2\ell^2}{\sin^2(2\pi h)}$$

that is

$$\ell^2 < \frac{h^2}{\delta^2(x)\sin^2(2\pi h) + x^2}$$

which holds by lemma 2.2.1. □

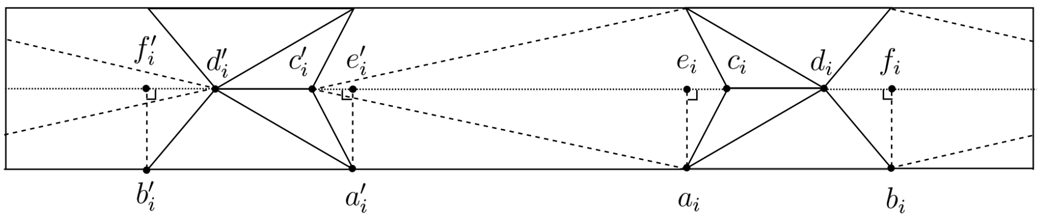

Figure 2.5: Constructions of e_i, f_i, e_i' and f_i'

2.4 Proof of Theorem 2.1.1

In this section, we prove that the map PL map $F : \mathbb{T}^2 \to \mathbb{E}^3$ described above is both isometric and an embedding. To show that F is isometric, it is enough to prove that every triangle of $\mathcal{T}(x,n)$ is mapped isometrically by F. In turn, this reduces to show that F preserves the lengths of the edges of the triangulation.
For practical reasons, we introduce supplementary families of points in $\mathcal{D}$. For every $i \in \{0, ..., n-1\}$ we consider the points $e_i \in [c_i c_i']$ such that $a_i c_i e_i$ is a right triangle (resp. $e_i' \in [c_i c_i']$ such that $a_i' c_i' e_i'$ is a right triangle). Similarly, we also consider the points $f_i \in [d_i d_i']$ such that $d_i b_i f_i$ is a right triangle (resp. $f_i' \in [d_i d_i']$ such that $d_i' b_i' f_i'$ is a right triangle), see Figure 2.5.

Lemma 2.4.1. *We have:*

$$\Omega_A = \left(0, 0, \frac{c_i c_i'}{2} - c_i e_i\right), \quad \Omega_B = \left(0, 0, \frac{d_i d_i'}{2} - d_i f_i\right)$$

and

$$\Omega_* = \left(0, 0, \frac{d_i d_i'}{2}\right) = \left(0, 0, \frac{c_i c_i'}{2}\right)$$

Proof. By construction $c_i d_i = \ell - e_i c_i - d_i f_i$, moreover $e_i c_i = \ell x$ and $d_i f_i = \ell y$ thus $c_i d_i = 0.1\ell$ and $1 - 2c_i d_i = 1 - 0.2\ell$. On the other hand, we have $1 - 2c_i d_i = d_i d_i' + c_i c_i' = 2d_i d_i' = 2c_i c_i'$ since $c_i c_i' = d_i d_i'$. Thus $\Omega_* = \left(0, 0, \frac{d_i d_i'}{2}\right) = \left(0, 0, \frac{c_i c_i'}{2}\right)$. The remaining cases are similar. □

Lemma 2.4.2. *Let*

$$\Psi(\ell, x, h) = \ell^2 \sigma(x) - \ell \left(r_4 - r_2 \cos(2\pi h)\right) \delta(x) \tag{2.4}$$

If ℓ satisfies $(A2)$ then $\Psi(\ell, x, h) = 0$.

Proof. Squaring $(A2)$ and rearranging we deduce

$$\ell^2 (\sigma(x) - (x + 0.1)\cos(2h\pi)\delta(x))^2 - \sin^2(2\pi h)\delta^2(x)(h^2 - x^2\ell^2) = 0.$$

It follows that

$$\ell(\sigma(x) - (x+0.1)\cos(2h\pi)\delta(x)) - \sin(2\pi h)\delta(x)\sqrt{h^2 - x^2\ell^2} = 0$$

or equivalently

$$\ell\sigma(x) - \delta(x)\left((x+0.1)\cos(2h\pi)\ell + \sin(2\pi h)\sqrt{h^2 - x^2\ell^2}\right) = 0. \tag{2.5}$$

We then observe that the rightmost factor is equal to $r_4 - r_2\cos\frac{\pi}{n}$, indeed

$$\begin{aligned} r_4 - r_2\cos\frac{\pi}{n} &= r_4 - r_4\cos^2\frac{\pi}{n} + \ell x\cos\frac{\pi}{n} + c_0d_0\cos\frac{\pi}{n} \\ &= r_4\left(1 - \cos^2\frac{\pi}{n}\right) + \ell\cos\frac{\pi}{n}(x+0.1) \\ &= r_4\sin^2\frac{\pi}{n} + \ell\cos\frac{\pi}{n}(x+0.1) \\ &= \sin\frac{\pi}{n}\sqrt{h^2 - x^2\ell^2} + \ell\cos\frac{\pi}{n}(x+0.1). \end{aligned}$$

We have obtained $\ell\sigma(x) - \delta(x)\left(r_4 - r_2\cos\frac{\pi}{n}\right) = 0$ □

Proposition 2.4.3. *The PL map $F : \mathbb{T}^2 \to \mathbb{E}^3$ is isometric.*

Proof of proposition 2.4.3. It is enough to prove that the distance of every edge $[p,q]$ in $\mathcal{T}(x,n)$ is preserved under f, i.e. $d_{\mathbb{E}^3}(F(p),F(q)) = d(p,q)$. To save space, we often write pq for $d(p,q)$ in this proof. A direct computation shows that $d^2_{\mathbb{E}^3}(F(c_i),F(c_i')) = d^2(c_i,c_i')$ and $d^2_{\mathbb{E}^3}(F(d_i),F(d_i')) = d^2(d_i,d_i')$. By lemma 2.4.1, we have

$$\begin{aligned} d^2_{\mathbb{E}^3}(F(a_i),F(b_i)) &= (r_4 - r_1)^2 + (c_ie_i - d_if_i)^2 \\ &= \ell^2\delta^2(x) + \ell^2(x-y)^2 = \ell^2 = (a_ib_i)^2 \end{aligned}$$

$$\begin{aligned} d^2_E(F(a_i),F(c_i)) &= r_4^2 + r_3^2 - 2r_4r_3\cos\frac{\pi}{n} + (c_ie_i)^2 \\ &= r_4^2 + \left(r_4\cos\frac{\pi}{n} - \ell x\right)^2 - 2r_4\left(r_4\cos\frac{\pi}{n} - \ell x\right)\cos\frac{\pi}{n} + (c_ie_i)^2 \\ &= r_4^2\sin^2\frac{\pi}{n} + \ell^2x^2 + (c_ie_i)^2 \\ &= h^2 + (c_ie_i)^2 = (a_ic_i)^2. \end{aligned}$$

$$\begin{aligned} d^2_{\mathbb{E}^3}(F(d_i),F(a_i)) &= r_4^2 + r_2^2 - 2r_4r_2\cos\frac{\pi}{n} + (c_ie_i)^2 \\ &= r_4^2 + \left(r_4\cos\frac{\pi}{n} - \ell x - c_id_i\right)^2 \\ &\quad - 2r_4\left(r_4\cos\frac{\pi}{n} - \ell x - c_id_i\right)\cos\frac{\pi}{n} + (c_ie_i)^2 \\ &= r_4^2\sin^2\frac{\pi}{n} + \ell^2x^2 + (c_id_i)^2 + 2c_id_i\ell x + (c_ie_i)^2 \\ &= h^2 + (c_id_i)^2 + 2c_id_i.c_ie_i + (c_ie_i)^2 \\ &= h^2 + (d_ic_i + c_ie_i)^2 = (a_id_i)^2. \end{aligned}$$

$$\begin{aligned}
d^2_{\mathbb{E}^3}(F(b_i), F(d_i)) &= r_1^2 + r_2^2 - 2r_1r_2\cos\frac{\pi}{n} + (d_if_i)^2 \\
&= r_2^2 + (r_4 - \ell\delta(x))^2 - 2r_2\left(r_4 - \ell\delta(x)\right)\cos\frac{\pi}{n} + (d_if_i)^2 \\
&= (d_if_i)^2 + r_2^2 + r_4^2 - 2r_2r_4\cos\frac{\pi}{n} \\
&+ \ell^2\delta^2(x) - 2r_4\ell\delta(x) + 2r_2\ell\delta(x)\cos\frac{\pi}{n}
\end{aligned}$$

Since $r_2^2 + r_4^2 - 2r_2r_4\cos\frac{\pi}{n} = d^2(F(d_i), F(a_i)) - (c_ie_i)^2$ we have

$$r_2^2 + r_4^2 - 2r_2r_4\cos\frac{\pi}{n} = h^2 + (d_ic_i + c_ie_i)^2 - (c_ie_i)^2 = h^2 + (d_ie_i)^2 - (c_ie_i)^2.$$

For the last equality we have used that e_i, c_i and d_i are aligned in this order. We then can write

$$\begin{aligned}
d^2_{\mathbb{E}^3}(F(b_i), F(d_i)) &= (d_if_i)^2 + h^2 + (d_ie_i)^2 - (c_ie_i)^2 + \ell^2\delta^2(x) \\
&- \ell\delta(x)\left(2r_4 - 2r_2\cos\frac{\pi}{n}\right) \\
&= (d_if_i)^2 + h^2 + \ell^2((1-y)^2 - x^2 + \delta^2(x)) \\
&- \ell\delta(x)\left(2r_4 - 2r_2\cos\frac{\pi}{n}\right) \\
&= (d_if_i)^2 + h^2 + 2\Psi(\ell, x, h).
\end{aligned}$$

For the second equality we have used that $d_ie_i = \ell(1-y)$ and $c_ie_i = \ell x$ and for the third equality we have used that $(1-y)^2 - x^2 + \delta^2(x) = 2\sigma(x)$. By lemma 2.4.2, $\Psi(\ell, x, h) = 0$, we thus have

$$d^2_{\mathbb{E}^3}(F(b_i), F(d_i)) = (b_id_i)^2$$

as desired. We also have

$$d^2_E(F(c_i), F(d_i)) = (r_3 - r_2)^2 = (c_id_i)^2.$$

By Lemma 2.4.1, we have

$$d_{\mathbb{E}^3}(F(a_i), F(a'_i)) = c_ic'_i - 2c_ie_i.$$

With the help of Figure 5, it is easily seen that

$$c_ic'_i - 2c_ie_i = c_ic'_i - c_ie_i - c'_ie'_i = a_ia'_i.$$

Similarly,

$$\begin{aligned}
d_{\mathbb{E}^3}(F(b_i), F(b'_i)) &= d_id'_i - 2d_if_i \\
&= d_id'_i - d_if_i - d'_if'_i \\
&= b_ib'_i.
\end{aligned}$$

Using once again Lemma 2.4.1 we compute

$$\begin{aligned} d^2_{\mathbb{E}^3}(F(a_i), F(c_i')) &= r_4^2 + r_3^2 - 2r_4 r_3 \cos\frac{\pi}{n} + (c_i c_i' - c_i e_i)^2 \\ &= h^2 + (c_i c_i' - c_i e_i)^2 \\ &= (a_i c_i')^2. \end{aligned}$$

A similar computation shows that $d^2_{\mathbb{E}^3}(F(a_{i+1}), F(c_i')) = d^2_{\mathbb{T}^2}(a_{i+1}, c_i')$. Finally,

$$\begin{aligned} d^2_{\mathbb{E}^3}(F(b_i), F(d_i')) &= r_1^2 + r_2^2 - 2r_1 r_2 \cos\frac{\pi}{n} + (d_i d_i' - d_i f_i)^2 \\ &= h^2 + (d_i d_i' - d_i f_i)^2 \\ &= (b_i d_i')^2 \end{aligned}$$

and similarly $d^2_{\mathbb{E}^3}(F(b_{i+1}), F(d_i')) = b_{i+1} d_i'$. □

Proposition 2.4.4. *The PL map $F : \mathbb{T}^2 \to \mathbb{E}^3$ is an embedding.*

Proof. The proof reduces to show that if the images of two triangles of $\mathcal{T}(x, n)$ have a non empty intersection set then they intersect along a common edge or vertex. Since the image of the ribbons are included in disjoint wedges, it is enough to check this fact for the eight triangles of a vertical ribbon of $\mathcal{T}(x, n)$. The computations related to this tedious check will not be reported here. □

Case of rectangular tori.– Elongating the vertical faces of $F(\mathbb{T}^2)$, we can obtain any rectangular flat torus.

Chapter 3

Linear embedding of the Moebius Torus

3.1 The Moebius' torus

We recall that a triangulation of the torus is given by a simplicial complex C and a homeomorphism $f : |C| \to \mathbb{T}^2$ where $|C|$ is the carrier space of C. It is well known that every surface can be triangulated.

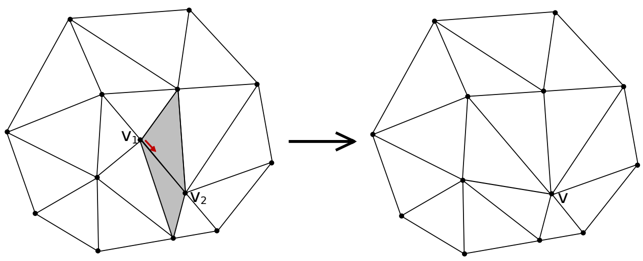

Figure 3.1: Edge contraction

A classical operation on triangulations is called **edge contraction**, it consists of collapsing an edge into a vertex (see figure 3.1). This operation does not always result in a simplicial complex. We say that a surface is **irreducible** if no edge can be contracted to produce a simplicial triangulation. It has been proved that every topological surface admits a finite number of irreducible triangulations [BE89]. In particular it is known that the torus has 21 irreducible triangulations [MT01, Sec. 5.4]. Among those the **Moebius' torus** is the only one with 7 vertices wich is minimal for all triangulations of the torus. More precisely the Moebius' torus has 7 vertices, 21 edges and 14 faces. The automorphism group has order 42 and is generated by $\tau = (0156324)$ and $\rho = (123456)$ (see figure 3.2).

Proposition 3.1.1. *If we number the vertices of* M $0,\ldots,6$ *so that the neighbors of vertex* 0 *are in the cyclic order* $1,2,3,4,5,6,1$*, then there are only two possible lists of triangles namely:*

(LI) : (012), (023), (034), (045), (056), (061), (135), (154), (142), (163), (264), (265), (253), (364).

(LII) : (012), (023), (034), (045), (056), (061), (143), (135), (125), (164), (254), (246), (263), (365).

The two possibilities correspond to the two possible orientations of the cycle of neighbors (see figure 3.2).

The first list correspond when the vertices are oriented clockwise and the second list correspond when they are oriented counterclockwise, we will take then the faces of the Moebius' torus as those in the list (LI). The list is invariant under cyclic permutation of indexes 1 to 6.

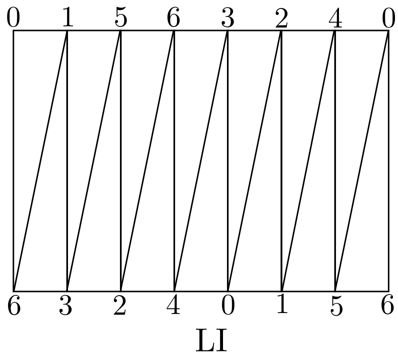

Figure 3.2: Combinatorial description of the Moebius' torus. The vertex numbering is such that the neighbors of vertex 0 appear ordered as $1,2,3,4,5,6,1$.

Note that the 1-skeleton of the Moebius' torus is the complete graph on seven vertices, which we denote by K_7. Although its size is very small, the Moebius' torus has many linear embeddings in $\mathbb{E}^3$ as proved in [BE], in particular the Csáczár torus is a linear embedding of the flat torus in $\mathbb{E}^3$ [Csá49] (see figure 3.3).
The main goal is to show that the number of vertices in the construction of a PL square flat torus made in the first part of this manuscript can not be lowered too much. For this we would like to show that the Moebius' torus has no linear embeddings isometric to the square flat torus in $\mathbb{E}^3$ and even more, in $\mathbb{E}^n$ with $n > 0$.
In this part we give strong arguments to conjecture that for M the Moebius' torus $\mathrm{L}_{\mathbb{T}^2}(M, \mathbb{E}^3)$ is empty, as we have already pointed, every element of

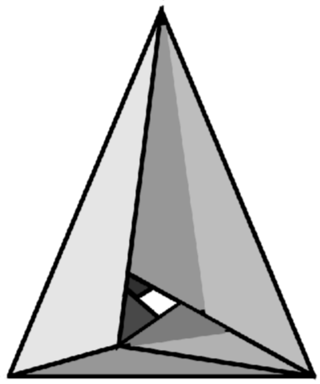

Figure 3.3: Csáczár torus: a linear embedding of $\mathbb{T}^2$ in $\mathbb{E}^3$.

$\mathrm{L}_{\mathbb{T}^2}(M, \mathbb{E}^3)$ can be seen as a linear isometrically embedding of a metric triangulation $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. We first give a precise definition of $\mathrm{GE}(M, \mathbb{T}^2)$ and we give some evidences that no $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$ can be isometrically linearly embedded in $\mathbb{E}^3$.
Before we consider this space, we first give some properties of the distance computation on $\mathbb{T}^2$. We note by $d_{\mathbb{E}^2}(,)$ the usual Euclidean distance in $\mathbb{E}^2$ and $d_{\mathbb{T}^2}(,)$ the distance in the square torus seen as a quotient of $\mathbb{E}^2$.
We consider $\pi : \mathbb{E}^2 \to \mathbb{T}^2$ the covering space of $\mathbb{T}^2$ with π the quotient map. For a point P in $\mathbb{E}^2$, we denote by

$$V_P := \left\{Q \in \mathbb{E}^2 | \ \forall P' \in \pi^{-1}\pi(P),\ d_{\mathbb{E}^2}(P, Q) \leq d_{\mathbb{E}^2}(P', Q)\right\}$$

its Voronoi cell with respect to $\pi^{-1}\pi(P) = P + \mathbb{Z}^2$. It is easily seen that $V_P = P + [-\frac{1}{2}, \frac{1}{2}]^2$. We will often use the following property:

Lemma 3.1.2. *Let $P, Q \in \mathbb{E}^2$, then $d_{\mathbb{E}^2}(P, Q) = d_{\mathbb{T}^2}(\pi(P), \pi(Q))$ if and only if $Q \in V_P$.*

Proof. Put $p = \pi(P)$ and $q = \pi(Q)$ and let γ a path from p to q. We denote by $\tilde{\gamma} : [0, 1] \to \mathbb{E}^2$ the lift of γ with $\tilde{\gamma}(0) = P$. We have $Q' := \tilde{\gamma}(1) \in P + \mathbb{Z}^2$ and we have

$$\mathrm{length}(\gamma) = \mathrm{length}(\tilde{\gamma}) \geq d_{\mathbb{E}^2}(P, Q') \geq d_{\mathbb{E}^2}(P, Q_0)$$

where Q_0 is any lift of q in V_P. The last inequality is strict whenever $Q' \notin V_P$. Note that all lifts of q in V_P are at the same distance from P. It follows that $\pi([P, Q_0])$ is a minimizing geodesic, whence $d_{\mathbb{T}^2}(p, q) = d_{\mathbb{E}^2}(P, Q_0)$. We conclude that $d_{\mathbb{T}^2}(p, q) = d_{\mathbb{E}^2}(P, Q)$ if and only if $Q \in V_P$. □

3.2 The space $GE(M, \mathbb{T}^2)$ of geodesic triangulations

Recall that a PL isometric embedding of $\mathbb{T}^2$ in $\mathbb{E}^3$ induces a geodesic triangulation $\mathcal{T}$ on $\mathbb{T}^2$ such that every edge of $\mathcal{T}$ is a geodesic segment in $\mathbb{T}^2$. In particular, every linear isometric embedding of M in $\mathbb{E}^3$ which is isometric to $\mathbb{T}^2$ induces a geodesic triangulation $\mathcal{T}$. In fact we claim that this triangulation satisfies the following **unique shortest path property**: every pair of vertices is connected by a unique geodesic in $\mathbb{T}^2$.

Proof of the claim: Consider a linear isometric embedding $f : M \to \mathbb{E}^3$. We denote by $\mathcal{T}$ the induced geodesic triangulation. Let p, q be two vertices on $\mathcal{T}$ and γ, λ be shortest paths from p to q (see figure 3.4). Then $f(\gamma)$ and $f(\lambda)$ are two shortest paths on $f(M) \subset \mathbb{E}^3$. Since the 1-skeleton of the Moebius' torus is a complete graph, pq is an edge of M such that $f(p)$ and $f(q)$ are connected by a line segment in $\mathbb{E}^3$. By uniqueness of shortest paths in $\mathbb{E}^3$ it follows that $f(\lambda) = f(\gamma) = [f(p), f(q)]$. Assume that γ and λ are different, we take the first point r where λ and γ become different. Clearly f cannot be injective on any neighborhood of r. This contradicts the fact that f is an immersion and so an embedding . □

Note that $\mathbb{E}^2$ acts isometrically by translations on the set of linear embeddings of M in $\mathbb{T}^2$. We denote by $\mathrm{GE}(M, \mathbb{T}^2)$ the space of linear embeddings of M in $\mathbb{T}^2$ with the shortest path property, where the embeddings are considered modulo the action of $\mathbb{E}^2$. It is equivalent to work with marked (minimizing) geodesic triangulations of $\mathbb{T}^2$ isomorphic to M. We will thus note by $\mathcal{T}$ an element of $\mathrm{GE}(M, \mathbb{T}^2)$ keeping in mind that the vertices of $\mathcal{T}$ have distinct labels. Hence, two triangulations are equivalent if and only if they are related by a translation that preserves the labels. We denote by $V(\mathcal{T}) = \{p_0, \ldots p_6\}$ the seven marked vertices of $\mathcal{T}$. From now on, we set $p_0 = \pi(0, 0)$ since $\mathcal{T}$ is taken modulo translations.

In this section, we study the space $\mathrm{GE}(M, \mathbb{T}^2)$. This space $\mathrm{GE}(M, \mathbb{T}^2)$ is endowed with a natural distance; for this we choose a basepoint $*$ of M once for all and given two embeddings $f, g : M \to \mathbb{T}^2$ we put $d(f, g) = \sup_x d_{\mathbb{T}^2}(f_*(x), g(x))$ where f_* is the translate of f that coincides with g at the basepoint $*$ of M.

Lemma 3.2.1. *d is a distance in $GE(M, \mathbb{T}^2)$ and the topology induced by d is independent of the base point of M.*

Proof. Let us first show that d is a distance: Take $d(f, h) = \sup_x d_{\mathbb{T}^2}(f_*(x), h(x))$. We have $d_{\mathbb{T}^2}(f_*(x), h(x)) = d_{\mathbb{T}^2}(f(x)+h(*)-f(*), h(x)) = d_{\mathbb{T}^2}(f(x)+h(*)-f(*) + g(*) - g(*), h(x)) = d_{\mathbb{T}^2}(f(x) + g(*) - f(*), h(x) + g(*) - h(*)) \leq d_{\mathbb{T}^2}(f(x) + g(*) - f(*), g(x)) + d_{\mathbb{T}^2}(g(x), h(x) + g(*) - h(*)) = d_{\mathbb{T}^2}(f(x) +$

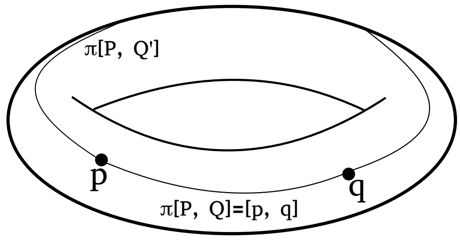

Figure 3.4: Two different paths form p q in $\mathbb{T}^2$.

$g(*)-f(*), g(x))+d_{\mathbb{T}^2}(g(x)-g(*)+h(*), h(x))$. This fact implies that for every $x \in M$, $\sup_x d_{\mathbb{T}^2}(f_*(x), h(x)) \leq \sup_x d_{\mathbb{T}^2}(f_*(x), g(x)) + \sup_x d_{\mathbb{T}^2}(g_*(x), h(x))$, or equivalently $d(f, h) \leq d(f, g) + d(g, h)$.

Now, $d(f, g) = \sup_x d_{\mathbb{T}^2}(f_*(x), g(x)) = \sup_x d_{\mathbb{T}^2}(f(x) - f(*) + g(*), g(x)) = \sup_x d_{\mathbb{T}^2}(f(x), g(x) - g(*) + f(*)) = d(g, f)$.

Moreover $d(f_*(x), g_*(x)) = 0$ if and only if $\sup_x d_{\mathbb{T}^2}(f_*(x), g_*(x)) = 0$ if and only if $f_*(x) = g_*(x)$.

We show now that the topology induced by d is independent of the base point $*$ of M. For that, take $d'(f, g) = \sup_x d_{\mathbb{T}^2}(f_{*'}(x), g(x))$ where $f_{*'}$ is the translated of f that coincides with g in another basepoint $*'$ of M. In general d and d' do not coincide but we have $d' < d < 2d$ so they define the same topology. We have $d_{\mathbb{T}^2}(f_*(x) + g_*(*') - f_*(*'), g_*(x)) = d_{\mathbb{T}^2}(f_*(x) - f_*(*'), g_*(x) - g_*(*')) \leq d_{\mathbb{T}^2}(f_*(x), g_*(x)) + d_{\mathbb{T}^2}(f_*(*'), g_*(*')) \leq 2d(f_*, g_*)$. □

Lemma 3.2.2. *Let $\mathcal{T} \in GE(M, \mathbb{T}^2)$ be a geodesic triangulation of $\mathbb{T}^2$, and let $[p, q]$ be an edge of $\mathcal{T}$. Then $[P, Q] \subset \mathbb{E}^2$ is a lift of $[p, q]$ if and only if $p = \pi(P)$, $q = \pi(Q)$, and $Q \in \mathring{V}_P$, i.e. $|x_Q - x_P| < \frac{1}{2}$ and $|y_Q - y_P| < \frac{1}{2}$, where $Q := (x_Q, y_Q)$ and $P = (x_P, y_P)$.*

Proof. Obviously we must have $p = \pi(P)$ and $q = \pi(Q)$ for $[P, Q]$ to be a lift of $[p, q]$. If $Q \in \mathring{V}_P$, then Q is the unique lift of q contained V_P. It follows from 3.1.2 that $\pi[P, Q] = [\pi(P), \pi(Q)]$ as desired. Now suppose that $[P, Q]$ is a lift of $[p, q]$. By lemma 3.1.2 we have $Q \in V_P$. If $Q \in \partial V_P$ then there exists $Q' \neq Q$ in V_P such that $q = \pi(Q')$. Then $\pi([P, Q'])$ and $\pi([P, Q])$ are two distinct shortest paths from p to q contradicting the unique shortest path property (see figure 3.4).

□

We define $D_0 := \left]-\frac{1}{2}, \frac{1}{2}\right]^2$ a fundamental domain of π. Note that the restriction of $\pi : D_0 \to \mathbb{T}^2$ is bijective, this is not true for V_P since for

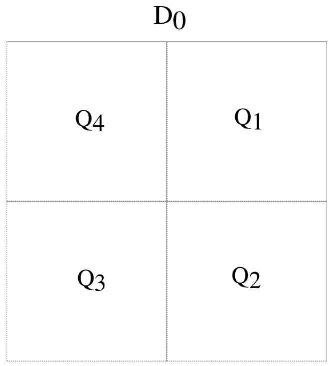

Figure 3.5: The four quadrants in the fundamental domain D_0.

every point Q in ∂V_P, there is at least another point $Q' \in \partial V_P$ such that $\pi(Q) = \pi(Q')$. Since D_0 is a fundamental domain of π, we can take the elements of $V(\mathcal{T})$ and work with their unique lift in D_0. We denote those lifts by $P_1, \dots, P_6$ (remember that $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$). Recall that the Moebius' torus induces an embedding of the complete graph K_7 in $\mathbb{T}^2$. This and the fact that $\mathcal{T}$ is given by a simplicial complex implies that the star StP_0 of the vertex P_0 consists of a cycle of six triangles and the link LkP_0 is composed of six edges. We may suppose that the labels of the vertices is such that they appear in the following (clockwise) order in the cycle: $P_1, P_2, P_3, P_4, P_5, P_6, P_1$. The points P_i can not be chosen independently, to see this we subdivide D_0 into four quadrants $Q_1 := \left]0, \frac{1}{2}\right] \times \left]0, \frac{1}{2}\right]$, $Q_2 := \left]0, \frac{1}{2}\right] \times \left]-\frac{1}{2}, 0\right]$, $Q_3 := \left]-\frac{1}{2}, 0\right] \times \left]-\frac{1}{2}, 0\right]$ and $Q_4 := \left]-\frac{1}{2}, 0\right] \times \left]0, \frac{1}{2}\right]$ (see figure 3.5).

We prove the following property:

Proposition 3.2.3. *Let $\mathcal{T} \in GE(M, \mathbb{T}^2)$. Every quadrant Q_i contains one or two vertices of $\tilde{\mathcal{T}}$. Moreover Q_i and Q_{i+2} have the same number of vertices for $i = 1, 2$.*

Remark: In particular the only possible distributions of the vertices are the two following ones: (see figure 3.6).

To prove proposition 3.2.3 we need the following lemmas:

Lemma 3.2.4. *Let $P, Q, R \in \mathbb{E}^2$ be joined by three segments in the lift of $\mathcal{T} \in GE(M, \mathbb{T}^2)$, then $[PQR]$ does not contain any lift of a vertex of $\mathcal{T}$ in its interior, in particular $[PQR]$ is a lift of a face of $\mathcal{T}$.*

Proof. Suppose that $[PQR]$ contains a lift of some vertex of $\mathcal{T}$, by lemma 3.1.2 the coordinates of P, Q and R differ by at most $\frac{1}{2}$. Hence $[PQR]$ lies in the interior of a fundamental domain of π. It ensues that $\pi([PQR])$ is a disc of $\mathbb{T}^2$, so that $\pi(\partial[PQR])$ is a separating 3-cycle in $\mathbb{T}^2$. Because K_7 has

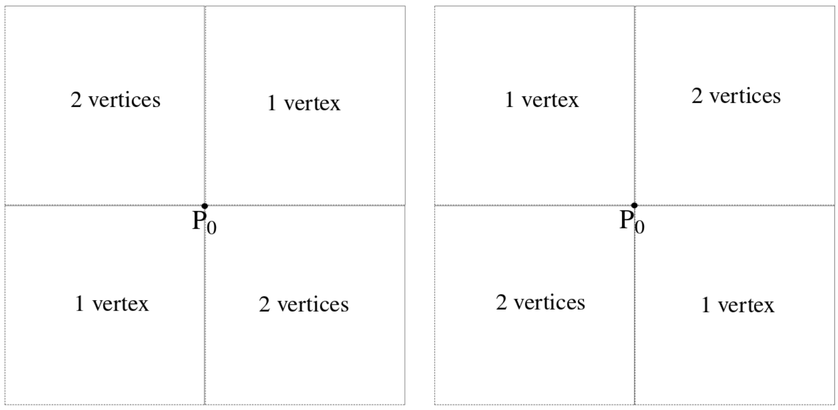

Figure 3.6: Possible arrangement of the vertices in each quadrants.

no separiting 3-cycle, it must be that all the remaining vertices lie inside $[PQR]$. However, this would contradict the fact that K_7 is not planar. □

Lemma 3.2.5. *A quadrant can not contain more than two vertices (P_0 not included).*

Proof. Assume there are three vertices P_1, P_2 and P_3 in the same quadrant Q_i. By lemma 3.2.2 $[P_i, P_j]$ must be a lift of the edge $[p_i, p_j]$ for $i, j \in \{0, 1, 2, 3\}$. It follows that the lift of the complete graph on $p_0, \ldots, p_3$ induces a complete graph K_4 on the points $P_0, \ldots, P_3$. Since the relative interior of the edges $[P_i, P_j]$ can not intersect, the only possible configurations for P_0, P_1, P_2 and P_3 is when some vertex P_i, $i \in \{1, 2, 3\}$, is in the interior of the triangle formed by the other three vertices P_j, P_k and P_0 (see figure 3.7). This is in contradiction with lemma 3.2.4. □

Remark 3.2.6. *There can not be three collinear vertices in Q_i. If this is the case then the edge between extrema points intersect with the edges between the medium point and every extrema point.*

Lemma 3.2.7. *Assume that Q_i contains two vertices (P_0 not included), then Q_{i+1} can not contain two vertices, (it is understood that $Q_{i+4} = Q_i$).*

Proof. Let P_j, P_{j+1} be two vertices in Q_i. Assume that Q_{i+1} contains two vertices. Recalling that the points are located in a cyclic clockwise order, those points must be P_{j+2} and P_{j+3}. We claim that $[P_j, P_{j+3}]$ is not an edge of the lift $\tilde{\mathcal{T}}$ of $\mathcal{T}$. Indeed if $[P_j, P_{j+3}]$ was an edge of $\tilde{\mathcal{T}}$, then by lemma 3.2.2, $[p_0 p_j p_{j+3}]$ would be a triangle of $\mathcal{T}$. However, according to our vertex

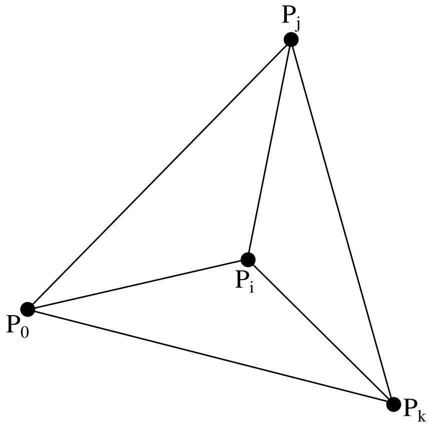

Figure 3.7: Configuration of points in Q_i.

indexation, the Moebius triangulation does not contain that triangle (see figure 3.2). By an analogous argument, we can claim that $[P_j, P_{j+2}]$ and $[P_{j+1}, P_{j+3}]$ are not edges of $\tilde{\mathcal{T}}$. Suppose that $i = 1$, then the previous claim implies that $[P_j, P_{j+3} + e_2]$, $[P_j, P_{j+3} + e_2]$ and $[P_{j+1}, P_{j+3} + e_2]$, (where e_1, e_2 is the canonical basis of $\mathbb{E}^2$) are edges of $\tilde{\mathcal{T}}$ so we have that $[P_j, P_{j+2} + e_2, P_{j+3} + e_2]$ and $[P_j, P_{j+3} + e_2, P_{j+1} + e_2]$ are triangles of $\tilde{\mathcal{T}}$ by lemma 3.1.2. We deduce that $[p_j, p_{j+2}, p_{j+3}]$ and $[p_j, p_{j+3}, p_{j+1}]$ must be triangles of $\mathcal{T}$. Whence $(j, j+2, j+3)$ and $(j, j+3, j+1)$ must be triangles of M. Looking at the two possible lists of faces in proposition 3.1.1, we conclude that this is not possible no matter the value of $j \in \{1, \ldots, 6\}$. For $i = 2, 3, 4$ are studied similarly replacing e_2 by $e_1, -e_2, -e_1$ respectively.

□

Proof of proposition 3.2.3

Let N_i the number of vertices in the quadrant Q_i without taking in account P_0. By lemma 3.2.5 we know that $\max(N_i) = 2$ for all $i = 1, \ldots, 4$. We have the three cases: $N_1 = 0$, $N_1 = 1$ and $N_1 = 2$.

- Case 1: $N_1 = 2$. By lemma 3.2.7, $N_2 \neq N_1$, so we have $N_2 = 0$ or $N_2 = 1$ by lemma 3.2.5. If $N_2 = 0$ then $N_3 + N_4 = 4$ which is impossible since at least one of Q_3 or Q_4 must contains three vertices contradicting lemma 3.2.5 or $N_3 = 2 = N_4$ contradiction lemma 3.2.7. Then $N_2 = 1$ necessarily. This implies that $N_3 + N_4 = 3$. By lemma3.2.5 Q_3 and Q_4 contain at least one vertex. Assume that $N_3 = 1$, then $N_4 = 2 = N_1$ which contradicts lemma 3.2.7. We have then necessarily $N_3 = 2$ and $N_4 = 1$. We are thus in the configuration described in the right side of figure 3.6.

- Case 2: $N_1 = 1$. If $N_2 = 1$, then $N_3 + N_4 = 4$ which is impossible by the same argument as in case 1. If $N_2 = 0$ then $N_3 + N_4 = 5$

but $\max(N_i) = 2$ then this case is also impossible. Necessary we have $N_2 = 2$. We are back to case 1 with N_2 replacing N_1, that means $N_3 = 1$ and $N_4 = 2$. We are thus in the configuration described in the left side of figure 3.6.

- Case 3: $N_1 = 0$. We have $N_2 + N_3 + N_4 = 6$. The only way of not contradicting lemma 3.2.5 if when $N_2 = N_3 = N_4 = 2$ but in this case lemma 3.2.7 is not verified. □

3.3 Configuration space

In this section we describe geometrically the space $\mathrm{GE}(M, \mathbb{T}^2)$. Note that if the vertices $V(\mathcal{T})$ in $\mathcal{T}$ are given, the triangulation $\mathcal{T}$ is entirely determined by the unique shortest path property and because the edges and the seven vertices should conform a complete graph K_7. We introduce the set $\tilde{V}(\mathcal{T})$, consisting of the lifts of vertices in $\mathcal{T}$ contained in D_0. There is clearly a bijection between $V(\mathcal{T})$ and $\tilde{V}(\mathcal{T})$. Indeed, if we denote by P_i the lift of p_i contained in D_0, P_i is unique because D_0 is a fundamental domain. The bijection is then given by $p_i \mapsto P_i$. Since P_0 is fixed to $(0, 0)$, the set $\tilde{V}(\mathcal{T})$ is determined by the 12 coordinates of $P_1, \ldots, P_6$. The above bijection allows us to identify $\mathrm{GE}(M, \mathbb{T}^2)$ with a certain subspace of $\mathbb{R}^{12}$. To study geometrically $\mathrm{GE}(M, \mathbb{T}^2)$, we will find different restrictions to the coordinates of $P_i \in \tilde{V}(\mathcal{T})$. We denote the coordinates of these vertices by $P_i = (x_i, y_i)$ for $i = 1, \ldots, 6$.

We denote by C_I the situation corresponding to elements of $\mathrm{GE}(M, \mathbb{T}^2)$ such that the lift of the elements of $V(\mathcal{T})$ are arranged as in the right side of figure 3.6. Similarly, the notation C_{II} corresponds to elements $\mathrm{GE}(M, \mathbb{T}^2)$ such that the lifts of the elements of $V(\mathcal{T})$ are arranged as in the left side of figure 3.6. Additionally we introduce the notation $C_{I,1}$ for the situation C_I such that the lift of p_1 is located in Q_2. The lift of p_1 is the only one between all the lifts of vertices on $\mathbb{T}^2$ located in Q_2 for the arguments given in the proof of lemma 3.2.3. In general, we denote by $C_{I,i}$ the case where C_I is such that the lift of p_i is the only one in Q_2 and by $C_{II,i}$ the case where C_{II} is such that the lift of p_i is the only one in Q_1 for $i = 1, \ldots, 6$. Recall that if there is a vertex P_i in Q_1 (or Q_2), since the vertices are ordered cyclically clockwise as $P_1, P_2, P_3, P_4, P_5, P_6, P_1$, all the quadrants of the vertices are determined as seen in lemma 3.6.

Remark 3.3.1. *Since the automorphisms of the map $p : \mathbb{R}^2 \to \mathbb{T}^2$ are the integer translations, then if $[P, Q]$ is a lift of $[p, q]$, then $[P + k_1 e_1 + k_2 e_2, Q + k_1 e_1 + k_2 e_2]$ is also a lift of $[p, q]$ for $k_i \in \mathbb{Z}, i = 1, 2$.*

Note also that if $[p, q, r]$ is a triangle of a triangulation $\mathcal{T}$ of $\mathbb{T}^2$, $[P, Q]$ a lift of $[p, q]$ and $[Q, R]$ a lift of $[q, r]$, then necessarily $[P, R]$ is a lift of $[p, r]$ such that $[P, Q, R]$ is a lift of the face $[p, q, r]$.

The following result give us necessary and sufficient conditions for the coordinates of P_i, for $i = 1, \ldots, 6$.

Theorem 3.3.2. *The subspace $C_{I,3} \subset GE(M, \mathbb{T}^2)$ is described by the following equations:*

(1). $x_2 - x_0 < \frac{1}{2}$

(2). $x_6 - x_2 < -\frac{1}{2}$

(3). $x_1 - x_6 < \frac{1}{2}$

(4). $x_4 - x_1 < -\frac{1}{2}$

(5). $x_3 - x_4 < \frac{1}{2}$

(6). $x_5 - x_3 < -\frac{1}{2}$

(7). $x_0 - x_5 < \frac{1}{2}$

and

(1'). $y_1 - y_0 < \frac{1}{2}$

(2'). $y_5 - y_1 < -\frac{1}{2}$

(3'). $y_6 - y_5 < \frac{1}{2}$

(4'). $y_3 - y_6 < -\frac{1}{2}$

(5'). $y_2 - y_3 < \frac{1}{2}$

(6'). $y_4 - y_2 < -\frac{1}{2}$

(7'). $y_0 - y_4 < \frac{1}{2}$

Before proving this theorem, we prove two consequences of the equations given in the preceding theorem. The first one gives us information about the position of the vertices $P_0, \ldots, P_6$. The second shows that the star around every vertex has the same structure, that is, StarP_i consists of a cycle of 6 triangles.

Lemma 3.3.3. *Let P_i verify equations 3.3.2, for $i = 0, \ldots, 6$, we have the following sequences:*

i) $x_5 < x_4 < x_6 < x_0 < x_3 < x_1 < x_2$

ii) $y_4 < y_3 < y_5 < y_0 < y_2 < y_6 < y_1$

Proof. i) Adding (5) and (6) we find $x_5 < x_4$, adding equations (3) and (4), we have $x_4 < x_6$. Adding (1) and (2) we have $x_6 < x_0 = 0$. Adding (6) and (7), we have $0 = x_0 < x_3$. Adding equations (4) and (5), we find $x_3 < x_1$. Finally, adding (2) and (3) we have $x_1 < x_2$. We verified the first sequence.

ii) Adding $(5')$ and $(6')$, we have $y_4 < y_3$. Adding $(3')$ and $(4')$, we have $y_3 < y_5$. Adding $(1')$ and $(2')$, we have $y_5 < y_0 = 0$, adding $(6')$ and $(7')$, we have $0 = y_0 < y_2$. Adding $(4')$ and $(5')$, we find $y_2 < y_6$. Finally, adding $(2')$ and $(3')$ we have $y_6 < y_1$.

□

The next lemma is a straightforward consequence of the equations given in 3.3.2:

Lemma 3.3.4. *Let $P_0, \ldots, P_6$ verifying the equations of theorem 3.3.2, with $P_0 = (0,0)$. Then $P_i \in D_0$ for all $i \in \{0, \ldots, 6\}$.*

Proof. We add equations (1) to (3) to find $x_1 < \frac{1}{2}$ and adding $-((4) + \cdots + (7))$ we have $x_1 > 0$. Then, we have

$$0 < x_1 < \frac{1}{2} \tag{3.1}$$

Equation $(1')$ and the addition of the negative of the remaining equations show that

$$0 < y_1 < \frac{1}{2} \tag{3.2}$$

Adding the negative of equations (2) to (7) and by equation (1), we find that

$$0 < x_2 < \frac{1}{2} \tag{3.3}$$

Adding $(1')$ to $(5')$ we find $y_2 < \frac{1}{2}$ and adding $-(6') - (7')$, we have $0 < y_2$. The above equations imply that

$$0 < y_2 < \frac{1}{2} \tag{3.4}$$

Adding (1) to (5) and making $-(6) - (7)$, we have

$$0 < x_3 < \frac{1}{2} \tag{3.5}$$

Adding $(1')$ to $(4')$ and adding the negative of the remaining equations, we have

$$-\frac{1}{2} < y_3 < 0 \tag{3.6}$$

Adding (1) to (4) and adding the opposite of the other equations, we have

$$-\frac{1}{2} < x_4 < 0 \tag{3.7}$$

By equation $(7')$ and the addition of equations $(1')$ to $(6')$ we find

$$-\frac{1}{2} < y_4 < 0 \tag{3.8}$$

Adding equations (1) to (6) and by equation (7), we have

$$-\frac{1}{2} < x_5 < 0 \tag{3.9}$$

Adding ($1'$) and ($2'$), and the addition of the negative of the other equations show that

$$-\frac{1}{2} < y_5 < 0 \tag{3.10}$$

Adding (1) and (2) and adding the negative of the remaining equations, we have

$$-\frac{1}{2} < x_6 < 0 \tag{3.11}$$

Adding equations ($1'$) to ($3'$), and adding the negative of the other equations, we have

$$0 < y_6 < \frac{1}{2} \tag{3.12}$$

Equations 3.1 to 3.12 (and the fact that $P_0 = (0, 0)$), imply $P_i \in D_0$ for all $i = 0, \ldots, 6$. □

Before proving the next result, we introduce some notations. We denote

$$D_i := P_i + D_0$$

and

$$Q_j^i := P_i + Q_j$$

for $i = 1, \ldots, 6$ and $j = 1, \ldots, 4$, so that D_i is a fundamental domain with P_i placed at the center of D_i and $\left\{Q_j^i\right\}_{j=1}^4$ are quadrants of D_i.

Lemma 3.3.5. *Let $P_0, P_1, \ldots, P_6$ contained in $\mathbb{R}^2$ with $P_0 = (0, 0)$, such that their coordinates verify the equations given in 3.3.2. The lift of the points $p_0, \ldots, p_6$ found in D_i define an hexagon along with a point inside it that we will call central point. Inside of this hexagon there is a cycle of six triangles, where all the triangles have the central point as common point and the interior of these triangles do not intersect. After an appropriate relabeling of the points $\{p_0, \ldots, p_6\}$, their lifts in D_i satisfy exactly the same equations as in 3.3.2 if the coordinate system is centered at P_i.*

Proof. First of all, note that by lemma 3.3.3, we can conclude that the points are disposed around $P_0 = (0, 0)$ in the cyclic order $P_1, P_2, P_3, P_4, P_5, P_6, P_1$.

We now verify that the lift of the points $p_0, \ldots, p_6$ in D_0 define the hexagon and the central point as described in the statement.

By lemma 3.3.4, $P_i \in D_0 \subset V_{P_0}$, we know that $|x_0 - x_i| < \frac{1}{2}$ and that $|y_0 - y_i| < \frac{1}{2}$. That means that $\pi([P_0, P_i])$ is the shortest path between $\pi(P_0) = p_0$ and $\pi(P_i) = p_i$. Therefore $[P_0, P_i]$ is a lift of the segment $[p_0, p_i]$ by lemma 3.2.2. We now verify that $\pi([P_i, P_{i+1}])$ is the shortest path between p_i and p_{i+1}:

- $\pi([P_1, P_2])$: Adding equations (2) and (3), we have $x_1 - x_2 < 0 < \frac{1}{2}$ and subtracting the remaining equations, i.e., making $-((4) + \cdots + (7)) - (1)$ we find $x_1 - x_2 > -\frac{1}{2}$, so $|x_1 - x_2| < \frac{1}{2}$. Similarly, if we add $(1') + (6') + (7')$ we have $y_1 - y_2 < \frac{1}{2}$ and if we subtract the rest of the equations, that is $-((2') + \cdots + (5'))$, we have $y_1 - y_2 > 0 > -\frac{1}{2}$. We deduce that $|y_1 - y_2| < \frac{1}{2}$, then $\pi([P_1, P_2])$ is the shortest path between p_1 and p_2.

- $\pi([P_2, P_3])$: Adding (1), (6) and (7) we have $x_2 - x_3 < \frac{1}{2}$, subtracting the remaining equations we have $x_2 - x_3 > 0 > -\frac{1}{2}$, so $|x_2 - x_3| < \frac{1}{2}$. On the other hand, from equation $(5')$ and subtraction of $(1') - (4')$ and $(6')$ and $(7')$ we have $-\frac{1}{2} < 0 < y_2 - y_3 < \frac{1}{2}$ so that $|y_2 - y_3| < \frac{1}{2}$, then $\pi([P_2, P_3])$ is the shortest path between p_2 and p_3.

- $\pi([P_3, P_4])$: From equation (5) and subtracting the reminding equations we find $\frac{1}{2} < 0 < x_3 - x_4 < \frac{1}{2}$. Making $(1') + \cdots + (4') + (7')$ and subtracting $(5')$ and $(6')$ we have $-\frac{1}{2} < 0 < y_3 - y_4 < \frac{1}{2}$. We deduce that $\pi([P_3, P_4])$ is the shortest path between p_3 and p_4.

- $\pi([P_4, P_5])$: Adding (1) to (4) and (7) and subtracting the remaining equations we obtain $-\frac{1}{2} < 0 < x_4 - x_5 < \frac{1}{2}$. Adding $(3')$ to $(6')$ and subtracting the other equations we have $-\frac{1}{2} < y_4 - y_5 < 0 < \frac{1}{2}$. That is, $|x_4 - x_5| < \frac{1}{2}$ and $|y_4 - y_5| < \frac{1}{2}$, thus $\pi([P_4, P_5])$ is the shortest path between $\pi(P_4) = p_4$ and $\pi(P_5) = p_5$.

- $\pi([P_5, P_6])$: Adding (3) to (6) and adding the negative of the remaining equations we have $-\frac{1}{2} < x_5 - x_5 < 0 < \frac{1}{2}$. On the other hand, if we add $(4')$ to $(7')$ and $(1') + (2')$, and by equation $(3')$. We have $-\frac{1}{2} < y_5 - y_6 < 0 < \frac{1}{2}$. This implies $|x_5 - x_6| < \frac{1}{2}$ and $|y_5 - y_6| < \frac{1}{2}$. We conclude that $\pi([P_5, P_6])$ is the shortest path between p_5 and p_6.

- $\pi([P_6, P_1])$: Adding equations (4) to (7) and (1) and (2), and taking the negative of equation (3), we obtain $-\frac{1}{2} < x_6 - x_1 < 0 < \frac{1}{2}$. Adding $(2')$ and $(3')$, and taking the negative of the addition of the remaining equations we have $-\frac{1}{2} < y_6 - y_1 < 0 < \frac{1}{2}$. Thus $|x_6 - x_1| < \frac{1}{2}$ and $|y_6 - y_1| < \frac{1}{2}$ implying that $\pi([P_6, P_1])$ is the shortest path between p_6 and p_1.

Since $\pi([P_0, P_i]), \pi([P_0, P_{i+1}])$ and $\pi([P_i, P_{i+1}])$ are the shortest path between their respective projections, we deduce from lemma 3.2.2 that $[P_0, P_i, P_{i+1}] \subset D_0$ is projected as a triangle $[p_0, p_i, p_{i+1}]$ contained in $\mathbb{T}^2$.

To fix the ideas, we see which are the lifts of p_i found in D_1. Evidently, P_1 is the lift of p_1 contained in D_1, In general, $P_j \in D_1$ if and only if $|x_1 - x_j| < \frac{1}{2}$ and $|y_1 - y_j| < \frac{1}{2}$, by definition of D_1.

- The lift of p_0 in D_1: Because $P_1 \in D_0$, we know that $|x_0 - x_1| < \frac{1}{2}$ and $|y_0 - y_1| < \frac{1}{2}$, therefore P_0 is the lift of p_0 contained in D_1.

- The lift of p_2 in D_1: As we proved previously, $\pi([P_1, P_2])$ is the shortest path between p_1 and p_2, that is $|x_1 - x_2| < \frac{1}{2}$ and $|y_1 - y_2| < \frac{1}{2}$. Hence, P_2 is the lift of p_2 in D_1.

- The lift of p_3 in D_1: Adding (1) to (3) and (6) + (7) and making $-(4) - (5)$, we find $-\frac{1}{2} < 0 < x_1 - x_3 < \frac{1}{2}$. Adding $(2')$ to $(4')$ and subtracting the other equations we have $\frac{1}{2} < y_1 - y_3 < 1$. Thus $-\frac{1}{2} < y_1 - (y_3 + 1) < 0 < \frac{1}{2}$, i. e., $|x_1 - x_3| < \frac{1}{2}$ and $|y_1 - (y_3 + 1)| < \frac{1}{2}$ and then $P_3 + e_2$ is the lift of p_3 in D_1.

- The lift of p_4 in D_1: Adding (1) to (3) to (5) to (7), and taking the negative of equation (4), we have $\frac{1}{2} < x_1 - x_4 < 1$ so that $-\frac{1}{2} < x_1 - (x_4 + 1) < 0 < \frac{1}{2}$. Adding equations $(1')$ and $(7')$, and adding the negative of the other equations, we find $\frac{1}{2} < y_1 - y_4 < 1$ and thus $-\frac{1}{2} < y_1 - (y_4 + 1) < \frac{1}{2}$, then $P_4 + e_1 + e_2$ is the lift of p_4 in D_1.

- The lift of p_5 in D_1: Adding (1) to (3) and (7), and making $-(4) - (5) - (6)$ we have $\frac{1}{2} < x_1 - x_5 < 1$ then $-\frac{1}{2} < x_1 - (x_5 + 1) < 0 < \frac{1}{2}$. Adding $(3')$ to $(7')$ and $(1')$ and by equation $-(2')$, we have $\frac{1}{2} < y_1 - y_5 < 1$ and then $-\frac{1}{2} < y_1 - (y_5 + 1) < 0 < \frac{1}{2}$, then $P_5 + e_1 + e_2$ is the lift of p_5 in D_1.

- The lift of p_6 in D_1: As we proved previously, $\pi([P_6, P_1])$ is the shortest path between p_6 and p_1, then P_6 is the lift of p_6 that is located in D_1.

We see now that the lifts of $p_0, \ldots, p_6$ in D_1 verify the equations given in 3.3.2; for that, we relabel these vertices: we denote $P_0^1 := P_1$, $P_1^1 := P_5 + e_1 + e_2$, $P_2^1 := P_4 + e_1 + e_2$, $P_3^1 := P_2$, $P_4^1 := P_0$, $P_5^1 := P_6$ and $P_6^1 := P_3 + e_2$. We note $P_i^1 = (X_i, Y_i)$.

By equation (4), we have $X_2 - X_0 = (x_4 + 1) - x_1 < -\frac{1}{2} + 1 = \frac{1}{2}$.
By equation (5), we have $X_6 - X_2 = x_3 - (x_4 + 1) < -\frac{1}{2}$.
By equation (6), we have $X_1 - X_6 = (x_5 + 1) - x_3 < \frac{1}{2}$.
By equation (7), we have $X_4 - X_1 = x_0 - (x_5 + 1) < -\frac{1}{2}$.
By equation (1), we have $X_3 - X_4 = x_2 - x_0 < \frac{1}{2}$.
By equation (2), we have $X_5 - X_3 = x_6 - x_2 < -\frac{1}{2}$.
By equation (3), we have $X_0 - X_5 = x_1 - x_6 < \frac{1}{2}$.
On the other hand, we have:
By equation $(2')$, we have $Y_1 - Y_0 = (y_5 + 1) - y_1 < \frac{1}{2}$.
By equation $(3')$, we have $Y_5 - Y_1 = y_6 - (y_5 + 1) < -\frac{1}{2}$.
By equation $(4')$, we have $Y_6 - Y_5 = (y_3 + 1) - y_6 < \frac{1}{2}$.
By equation $(5')$, we have $Y_3 - Y_6 = y_2 - (y_3 + 1) < -\frac{1}{2}$.
By equation $(6')$, we have $Y_2 - Y_3 = (y_4 + 1) - y_2 < \frac{1}{2}$.
By equation $(7')$, we have $Y_4 - Y_2 = y_0 - (y_4 + 1) < -\frac{1}{2}$.
By equation $(1')$, we have $Y_0 - Y_4 = y_1 - y_0 < \frac{1}{2}$.

Since the points P_i^1 verify the same equations as the points P_i, then the set of points $\left\{P_i^1\right\}_{i=0}^{6}$ verify the same properties of the set of points

$\{P_i\}_{i=0}^6$, that is, the lifts of $p_0, \ldots, p_6$ in D_1 conform an hexagon with edges $[P_i^1, P_{i+1}^1]$, along with a central point (in this case the central point is P_1) and inside the hexagon six triangles $[P_i^1, P_0^1, P_{i+1}^1]$. Also, the points appear in a cyclic order around P_1 as follows: $P_1^1, P_2^1, P_3^1, P_4,^1 P_5^1, P_6^1, P_1^1$.

Similarly, it can be shown that the lifts of $p_0, \ldots, p_6$ in D_2 conform an hexagon along with the central point P_2. These lifts are placed as follows: we relabel as $P_0^2 := P_2$, $P_1^2 := P_4 + e_1 + e_2$, $P_2^2 := P_6 + e_1$, $P_3^2 := P_5 + e_1$, $P_4^2 := P_3$, $P_5^2 := P_0$ and $P_6^2 := P_1$ so that $P_3^2 \in Q_2^2$ and it is the only lift in this quadrant. As in the previous case, the coordinates of the points $\left\{P_i^2\right\}_{i=0}^6$ verify the equations 3.3.2.

For the lifts of $\{p_0, \ldots, p_6\}$ contained in D_3, the relabeling is made as follows: $P_0^3 := P_3$, $P_1^3 := P_2$, $P_2^3 := P_5 + e_1$, $P_3^3 := P_1 - e_2$, $P_4^3 := P_6 - e_2$, $P_5^3 := P_4$ and $P_6^3 := P_0$. It can be shown that P_3^3 is the only point in Q_2^3. The coordinate of this points verify equations 3.3.2.

For the lifts of $\{p_0, \ldots, p_6\}$ contained in D_4, the relabeling is made as follows: $P_0^4 := P_4$, $P_1^4 := P_0$, $P_2^4 := P_3$, $P_3^4 := P_6 - e_2$, $P_4^4 := P_2 - e_1 - e_2$, $P_5^4 := P_1 - e_1 - e_2$ and $P_6^4 := P_5$. It can be shown that P_3^4 is the only point in Q_2^4. The coordinate of this points verify equations 3.3.2.

For the lifts of $\{p_0, \ldots, p_6\}$ contained in D_5, the relabeling is made as follows: $P_0^5 := P_5$, $P_1^5 := P_6$, $P_2^5 := P_0$, $P_3^5 := P_4$, $P_4^5 := P_1 - e_1 - e_2$, $P_5^5 := P_3 - e_1$ and $P_6^5 := P_2 - e_1$. It can be shown that P_3^5 is the only point in Q_2^5. The coordinate of this points verify equations 3.3.2.

For the lifts of $\{p_0, \ldots, p_6\}$ contained in D_6, the relabeling is made as follows: $P_0^6 := P_6$, $P_1^6 := P_3 + e_2$, $P_2^6 := P_1$, $P_3^6 := P_0$, $P_4^6 := P_5$, $P_5^6 := P_2 - e_1$ and $P_6^6 := P_4 + e_2$. It can be shown that P_3^6 is the only point in Q_2^6. The coordinate of this points verify equations 3.3.2.

For all the lifts of $\{p_0, \ldots, p_6\}$ in D_i for $i = 2, \ldots, 6$ it can be shown as in the case D_1, that the lifts verifies the same equations and therefore the same properties as in the case D_0. □

We recap the notations for P_j^i in the following dictionary.

Definition 3.3.6 (Dictionary)**.**

$$P_0^i := P_i \quad \textit{and} \quad P_i^0 := P_i \quad \textit{for} \quad i = 0, \ldots, 6$$

.

$P_1^1 := P_5 + e_1 + e_2$ $P_4^1 := P_0$	$P_2^1 := P_4 + e_1 + e_2$ $P_5^1 := P_6$	$P_3^1 := P_2$ $P_6^1 := P_3 + e_2$
$P_1^2 := P_4 + e_1 + e_2$ $P_4^2 := P_3$	$P_2^2 := P_6 + e_1$ $P_5^2 := P_0$	$P_3^2 := P_5 + e_1$ $P_6^2 := P_1$
$P_1^3 := P_2$ $P_4^3 := P_6 - e_2$	$P_2^3 := P_5 + e_1$ $P_5^3 := P_4$	$P_3^3 := P_1 - e_2$ $P_6^3 := P_0$
$P_1^4 := P_0$ $P_4^4 := P_2 - e_1 - e_2$	$P_2^4 := P_3$ $P_5^4 := P_1 - e_1 - e_2$	$P_3^4 := P_6 - e_2$ $P_6^4 := P_5$
$P_1^5 := P_6$ $P_4^5 := P_1 - e_- e_2$	$P_2^5 := P_0$ $P_5^5 := P_3 - e_1$	$P_3^5 := P_4$ $P_6^5 := P_2 - e_1$
$P_1^6 := P_3 + e_2$ $P_4^6 := P_5$	$P_2^6 := P_1$ $P_5^6 := P_2 - e_1$	$P_3^6 := P_0$ $P_6^6 := P_4 + e_2$

Recall that from the proof of lemma 3.3.5, the points P_j^i play the same role as the point P_j in equations 3.3.2, when we are placed in D_i. Then, all properties that are true for P_j are also true for P_j^i.

Remark 3.3.7. *A consequence of lemma 3.3.5 is that there is one and only one lift of p_j contained in Q_2^i for all $i = 0, \ldots, 6$*

Proof of theorem 3.3.2. $\Leftarrow$) Suppose that equations (1) to (7) and (1′) to (7′) are verified. We will prove that for every $\pi(P_i)$ with $i = 1, \ldots, 6$;

A. There is uniqueness of shortest path between every pair of points.

B. The set of these shortest paths induce a triangulation of the torus corresponding to the configuration $C_{I,3}$.

From lemma 3.3.4, we know that $P_i \in D_0$ for all $i = 0, \ldots, 6$. We can say even more; equations 3.1 to 3.12, imply that $P_1, P_2 \in Q_1$, $P_3 \in Q_2$, $P_4, P_5 \in Q_3$ and $P_6 \in Q_4$ so we are in a configuration $C_{I,3}$. Moreover, by lemma 3.3.3 the vertices P_i are found in a cyclic clockwise order $P_1, P_2, P_3, P_4, P_5, P_6, P_1$.

We verify now that the points $P_0, \ldots, P_6$ induce a triangulation $\tilde{\mathcal{T}}$ and consequently their projections $\pi(P_i)$ induce a triangulation by geodesic segments $\mathcal{T}$ of $\mathbb{T}^2$.

As proved in lemma 3.3.5, around P_0 there is a cycle of six triangles $[P_0, P_i, P_{i+1}]$ where the projection of every segment is the shortest path between the respective vertices. We know that the vertices p_i define a complete graph K_7, so there are in all 21 edges. Because the triangles $[P_0, P_i, P_{i+1}]$ define 12 edges, and we know that these edges do not intersect, it remains to prove that the remaining 9 edges do not intersect with any lift of the

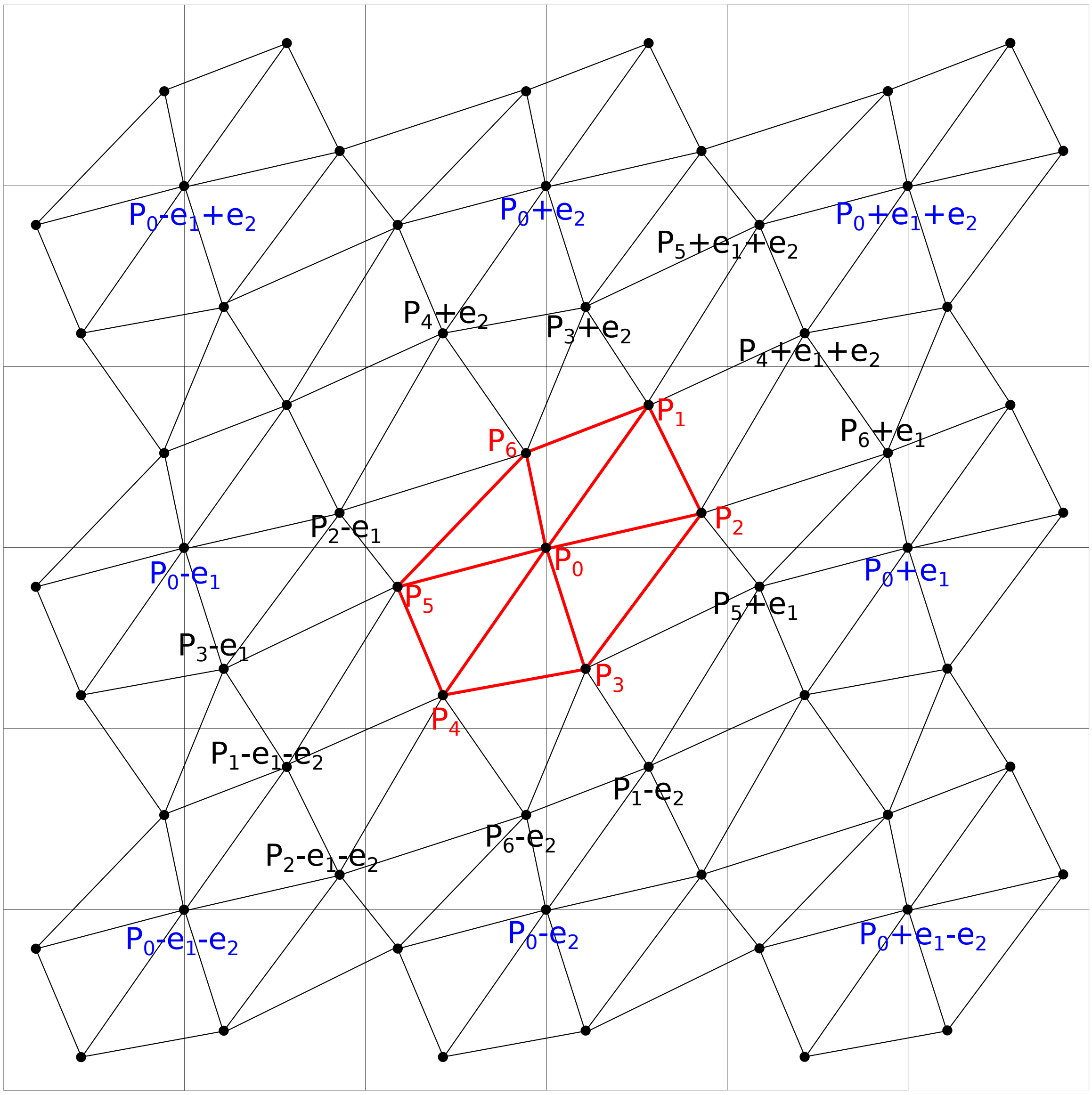

Figure 3.8: Tiling of $\mathbb{E}^2$ with the lift of a triangulation of $\mathbb{T}^2$.

hexagon defined by the lifts of p_i, for $i = 0, \ldots, 6$ in $D_0 + ae_1 + ae_2$ for all $(a, b) \in \mathbb{Z}^2$.

First of all, we show which one is the shortest path between the vertices corresponding to the remaining edges:

- Shortest path between p_1 and p_3: As we saw in the proof of 3.3.5, $P_3 + e_2$ is the lift of p_3 in D_1, then $\pi([P_3 + e_2, P_1])$ is the shortest path between p_3 and p_1.

- Shortest path between p_1 and p_4: As we saw in the proof of 3.3.5, $P_4 + e_1 + e_2$ is the lift of p_4 in D_1, then $\pi([P_4 + e_1 + e_2, P_1])$ is the shortest path between p_4 and p_1.

- Shortest path between p_1 and p_5: As we saw in the proof of 3.3.5, $P_5 + e_1 + e_2$ is the lift of p_5 in D_1, then $\pi([P_5 + e_1 + e_2, P_1])$ is the shortest path between p_5 and p_1.

- Shortest path between p_2 and p_4: Adding equations (2) to (4), we have $(x_4 + 1) - x_2 < \frac{1}{2}$ and adding the remaining equations we find $-\frac{1}{2} < - < (x_4 + 1) - x_2$. On the other hand, by equation $(6')$, we conclude that $-\frac{1}{2} < y_2 - (y_4 + 1)$ and adding the remaining equations we find $y_2 - (y_4 + 1) < 0 < \frac{1}{2}$. Then $|x_2 - (x_4 + 1)| < \frac{1}{2}$ and $|y_2 - (y_4 + 1)| < \frac{1}{2}$, then $\pi([P_4 + e_1 + e_2, P_2])$ is the shortest path between p_4 and p_2.

- Shortest path between p_2 and p_5: Adding (2) to (6), and adding the negative of the remaining equations, we find $-1 < x_5 - x_2 < -\frac{1}{2}$ then $-\frac{1}{2} < (x_5 + 1) - x_2 < \frac{1}{2}$. If we add $(1')$, $(2')$, $(6')$ and $(7')$ and adding the negative of the other equations, we find $-\frac{1}{2} < y_5 - y_2 < 0 < \frac{1}{2}$, then $|(x_5 + 1) - x_2| < \frac{1}{2}$ and $|y_5 - y_2| < \frac{1}{2}$ so $\pi([P_2, P_5 + e_1])$ is the shortest path between p_2 and p_5.

- Shortest path between p_2 and p_6: By equation (2), we have $\frac{1}{2} < x_2 - x_6$ and adding the remaining equations, we have $x_2 - x_6 < 1$. That means $-\frac{1}{2} < x_2 - (x_6 + 1) < \frac{1}{2}$. Adding $(4')$ and $(5')$ and adding the negative of the other equations, we have $-\frac{1}{2} < y_2 - y_6 < 0$ or equivalently $-\frac{1}{2} < y_2 - y_6 < \frac{1}{2}$. Then $\pi([P_2, P_6 + e_1])$ is the shortest path between p_2 and p_6.

- Shortest path between p_3 and p_5: Adding (1) to (5) and (7) and taking $-(6)$, we have $\frac{1}{2} < x_3 - x_5 < 1$ so that $-\frac{1}{2} < x_3 - (x_5 + 1) < 0 < \frac{1}{2}$. On the other hand, if we add $(3')$ and $(4')$ and we add the negative of the remaining equations we find $-\frac{1}{2} < y_3 - y_5 < 0 < \frac{1}{2}$. That is $|x_3 - (x_5 + 1)| < \frac{1}{2}$ and $|y_3 - y_5| < \frac{1}{2}$ We conclude that $\pi([P_5 + e_1, P_3])$ is the shortest path between p_5 and p_3.

- Shortest path between p_3 and p_6: Adding equations $(3), (4)$ and (5), we have $x_3 - x_6 < \frac{1}{2}$. Adding the remaining equations we have $-\frac{1}{2} <$

$x_3 - x_6$. On the other hand, equation (4′) tell us that $y_3 - (y_6 - 1) < \frac{1}{2}$ and adding the other equations we find $-\frac{1}{2} < 0 < y_3 - (y_6 - 1)$. Then $|x_3 - x_6| < \frac{1}{2}$ and $|y_3 - (y_6 - 1)| < \frac{1}{2}$. This implies that $\pi([P_3, P_6 - e_2])$ is the shortest path between p_3 and p_6

- Shortest path between p_4 and p_6: Adding (1), (2), and (5) to (7) and adding the negative of (3) and (4), we have $-\frac{1}{2} < 0 < x_6 - x_4 < \frac{1}{2}$. Adding (1′) to (3′) and (7′) and adding the negative of the remaining equations, we have $\frac{1}{2} < y_6 - y_4 < 1$ so $-\frac{1}{2} < (y_6 - 1) - y_4 < \frac{1}{2}$. We conclude that $\pi([P_4, P_6 - e_2])$ is the shortest path between p_4 and p_6.

Once we know which are the lifts of the shortest paths between the vertices, we prove that in fact they do not intersect any lift of $[p_i, p_j]$ in $D_0 + ae_1 + be_2$ for all $(a, b) \in \mathbb{Z}^2$.

1. $[P_1, P_3 + e_2]$: The lift of $[p_1, p_3]$ is $[P_1, P_3 + e_2]$. This edge intersects D_0 and $D_0 + e_2$ so we will verify that $[P_1, P_3 + e_2]$ does not intersect the hexagon in D_0 or in $D_0 + e_2$ defined by the lifts of $\{p_i\}_{i=0}^{6}$. By lemma 3.3.3 and 3.3.2, we know that $x_3 < x_1$ and $y_1 < y_3 + e_2$. Moreover, every point $Q := (x, y) \in [P_1, P_3 + e_2]$ verifies $x_3 < x < x_1$ and $y_1 < y < y_3 + 1$. If $[P_1, P_3 + e_1]$ intersects an edge $[P_i, P_j]$ in $D_0 + e_2$, that would mean that there is a point $R := (u, v) \in [P_i, P_j]$ such that $x_3 < u < x_1$ and $y_1 < v < y_3 + e_2$. Now by lemma 3.3.3, all points $S = (\alpha, \beta)$ contained in $[P_0 + e_2, P_3 + e_2]$, $[P_3 + e_2, P_2 + e_2]$, $[P_0 + e_2, P_2 + e_2]$, $[P_1 + e_2, P_2 + e_2]$, $[P_0 + e_2, P_1 + e_2]$, $[P_0 + e_2, P_6 + e_2]$, $[P_6 + e_2, P_1 + e_2]$, $[P_5 + e_2, P_6 + e_2]$, and $[P_0 + e_2, P_5 + e_2]$ verify that $\beta > y_3 + 1$ and consequently any point in these edges could be contained also in $[P_1, P_3 + e_2]$. On the other hand, any point $T := (t_1, t_2)$ contained in $[P_4 + e_2, P_0 + e_2]$, $[P_4 + e_2, P_3 + e_2]$ or $[P_4 + e_2, P_5 + e_2]$, verifies by lemma 3.3.3, that $t_1 < x_3$, then any point in these edges could be also contained in $[P_1, P_3 + 1]$. We conclude that $[P_1, P_3 + e_2]$ does not intersect any edge of the hexagon in $D_0 + e_2$. Remark also that by lemma 3.3.3, we know that every point $U := (u_1, u_2)$ on the hexagon contained in D_0 verifies that $u_2 < y_1$. This implies that any point in this hexagon but P_1 can be contained in $[P_1, P_3 + e_2]$. We deuce that $[P_1, P_3 + e_2]$ does not intersect any edge of the hexagon in D_0 neither. Then $[P_1, P_3 + e_2]$ does not intersect any hexagon in $D_0 + ae_1 + be_2$ for all $a, b \in \mathbb{Z}$.

2. $[P_3 + e_2, P_6]$: The lift of $[p_3, p_6]$ as seen previously is $[P_3 + e_2, P_6]$. As in the previous case, it is sufficient to verify that $[P_3 + e_2, P_6]$ does not intersect any lift of $[p_i, p_j]$ in D_0 and in $D_0 + e_2$. By lemma 3.3.3, we know that any point $Q := (x, y)$ contained in $[P_3 + e_2, P_6]$ verifies that $x_6 < x < x_3$ and $y_6 < y < y_3 + 1$. As in the previous case, every point $S := (s_1, s_2)$ different of $P_3 + e_2$, contained in $[P_0 + e_2, P_3 + e_2]$,

$[P_3+e_2, P_2+e_2]$, $[P_0+e_2, P_2+e_2]$, $[P_1+e_2, P_2+e_2]$, $[P_0+e_2, P_1+e_2]$, $[P_0+e_2, P_6+e_2]$, $[P_6+e_2, P_1+e_2]$, $[P_5+e_2, P_6+e_2]$, and $[P_0+e_2, P_5+e_2]$ verifies that $s_2 > y_1 + 1$. That means that any point contained in these edges can be also contained in $[P_3 + e_2, P_6]$. Similarly, any point $T := (t_1, t_2)$ different to P_6 contained in $[P_6, P_0]$, $[P_5, P_0]$, $[P_4, P_0]$, $[P_3, P_0]$, $[P_2, P_0]$, $[P_6, P_5]$, $[P_5, P_4]$, $[P_4, P_3]$ and $[P_3, P_2]$ verifies that $t_2 < y_6$, therefore any point in these edges could be also be contained in $[P_6, P_3+e_2]$. Now, the lines (P_4+e_2, P_3+e_2) and (P_6, P_3+e_2) intersect in P_3+e_2 and since $y_6 < y_4+1$ then $P_6 \neq P_4+e_2$, therefore, these lines can not intersect in any other point, we can say even more, $P_4 + e_2$ is strictly over P_6, then the segment $[P_4 + e_2, P_3 + e_2]$ is completely over the segment $[P_6, P_3 + e_2]$ outside their common point point $P_3 + e_2$. Similarly, we can deduce that $[P_6, P_1]$ is completely under the segment $[P_6, P_3+e_2]$ except for the point P_6 where they intersect. We conclude that $[P_6, P_3 + e_2]$ does not intersect $[P_1, P_6]$ or $[P_4 + e_2, P_3 + e_2]$. Now, every point $U : (u_1, u_2)$ in $[P_4 + e_2, P_5 + e_2]$ verifies that $u_1 < x_4$ and by lemma 3.3.3 we know that $x_4 < x_6$, then any point contained in this segment could be also contained in $[P_6, P_3+e_2]$. Every point $V := (v_1, v_2)$ contained in $[P_1, P_2]$ verifies by lemma 3.3.3 that $v_1 > x_1 > x_3$ so any point on this segment could be also be in the segment $[P_6, P_3 + e_2]$. Finally, we verify that $[P_6, P_3 + e_2]$ can not intersect $[P_1, P_0]$ or $[P_4 + e_2, P_0 + e_2]$. Suppose that $[P_6, P_3 + e_2]$ touches $[P_4 + e_2, P_0 + e_2]$. since every point $W := (w_1, w_2)$ in $[P_4 + e_2, P_0 + e_2]$ verifies by lemma 3.3.3 that $x_4 < w_1 < x_3$ and $y_4 + 1 < w_2 < y_0 + 1$, then outside of the point $P_4 + e_2$, the segment $[P_4 + e_2, P_0 + e_2]$ is completely over the segment $[P_4 + e_2, P_3 + e_2]$, and as we show previously, $[P_6, P_3 + e_2]$ is completely under the segment $[P_4 + e_2, P_3 + e_2]$. If there is an intersection point between $[P_6, P_3 + e_2]$ and $[P_4 + e_2, P_0 + e_2]$, then there exist a point $I := (\alpha, \beta)$ such that $\alpha < x_3$ and $I \in [P_6, P_3 + e_2] \cap [P_4 + e_2, P_3 + e_2]$, bu the intersection point of these segments was $P_3 + e_2$ so there is a contradiction. A similar argument shows that $[P_6, P_3 + e_2]$ and $[P_1, P_0]$ can not intersect since $[P_1, P_0]$ is completely under $[P_6, P_1]$ (outside the point P_1) and every point $Z := (z_1, z_2)$ in $[P_1, P_0]$ verifies $x_6 < z_1 < x_1$. We conclude that $[P_6, P_3 + e_2]$ does not intersect any edge of the hexagons in D_0 or in $D_0 + e_2$, therefore any hexagon in $D_0 + ae_1 + be_2$ for all $a, b \in \mathbb{Z}$.

3. $[P_4 + e_2, P_6]$: The lift of $[p_4, p_6]$ is $[P_4 + e_2, P_6]$. Since this segment is contained in D_0 and in $D_0 + e_2$, we verify that it does not intersect any edge of the hexagons in D_0 or in $D_0 + e_2$. Every point $Q := (x, y)$ contained in $[P_4+e_2, P_6]$ verifies that $x_4 < x < x_6$ and $y_6 < y < y_4+1$ by lemma 3.3.3 and equations 3.3.2. Now, by the same lemma, we have that every point $R = (r_1, r_2)$ in $[P_6, P_0]$, $[P_6, P_5]$, $[P_0, P_5]$, $[P_5, P_4]$, $[P_0, P_4]$, $[P_4, P_3]$, $[P_0, P_3]$, $[P_3, P_2]$ and $[P_0, P_2]$ verifies that $r_2 < y_6$ then

any point in these segments could be also contained in $[P_6, P_4 + e_2]$. Moreover every point $S := (s_1, s_2)$ conforming the hexagon and the edges $[P_0 + e_2, P_i + e_2]$ for $i = 1, \ldots, 6$ verify that $s_2 > y_4 + 1$ then $[P_4 + e_2, P_6]$ can not intersect any lift of $[p_i, p_j]$ in $D_0 + e_2$. Now, every point $T := (t_1, t_2)$ in $[P_1, P_0]$, $[P_1, P_6]$ and $[P_1, P_2]$ verifies that $t_1 > x_6$, then any point in these edges can be also in the edge $[P_6, P_4 + e_2]$. This and the previous arguments shows that the edge $[P_6, P_4 + e_2]$ does not intersect the hexagon in D_0 nor in $D_0 + e_2$. In consequence, it does not intersect any lift of $[p_i, p_j]$ in $D_0 + ae_1 + be_2$ for all $A, B \in \mathbb{Z}$.

4. $[P_2, P_5 + e_1]$: The lift of $[p_2, p_5]$ is $[P_2, P_5 + e_1]$. By lemma 3.3.3, we know that any point $Q := (x, y)$ in $[P_2, P_5 + e_1]$ verifies that $x_2 < x < x_5 + 1$ and that $y_5 < y < y_2$. In one hand, every point $R := (r_1, r_2)$ in the lifts of $[p_i, p_j]$ in D_0 verify that $r_1 < x_2$. On the other hand, every point $S := (s_1, s_2)$ contained in the lifts of $[p_i, p_j]$ in $D_0 + e_1$ verifies that $s_1 > x_5 + 1$, then it could not be also in $[P_2, P_5 + e_1]$. Then, $[P_2, P_5 + e_1]$ does not intersect any lift of $[p_i, p_j]$ in D_0 or in $D_0 + e_1$ and consequently in $D_0 + ae_1 + be_2$ for all $a, b \in \mathbb{Z}$.

5. $[P_2, P_6 + e_1]$: The lift of $[p_2, p_6]$ is $[P_2, P_6 + e_1]$. By lemma 3.3.3 and equations 3.3.2, we know that every point $Q := (x, y) \in [P_2, P_6 + e_1]$ is such that $x_2 < x < x_6 + 1$ and $y_2 < y < y_6$. Now, every point $R = (r_1, r_2)$ outside of P_2 contained in the lifts of $[p_i, p_j]$ in D_0 is such that $r_1 < x_2$, thus neither of these points can be contained also in $[P_2, P_6 + e_1]$. This implies that $[P_2, P_6 + e_1]$ does not intersect the hexagon in D_0. On the other hand, all the points $S := (s_1, s_2)$ outside $P_6 + e_1$ contained in the edges $[P_6 + e_1, P_1 + e_1]$, $[P_6 + e_1, P_0 + e_1]$, $[P_1 + e_1, P_0 + e_1]$, $[P_1 + e_1, P_2 + e_1]$, $[P_2 + e_1, P_0 + e_1]$, $[P_2 + e_1, P_3 + e_1]$, $[P_3 + e_1, P_0 + e_1]$, $[P_3 + e_1, P_4 + e_1]$,and $[P_4 + e_1, P_0 + e_1]$ verify that $s_1 > x_6 + 1$, then any point in these edges could be also contained in $[P_2, P_6 + e_1]$. Therefore, $[P_2, P_6 + e_1]$ does not intersect any of these edges. Now, since $y_0 < y_2$ by lemma 3.3.3, every point $T := (t_1, t_2)$ contained in $[P_5 + e_1, P_0 + e_1]$ and in $[P_5 + e_1, P_4 + e_1]$ verifies that $t_2 < y_2$ by lemma 3.3.3, then there are no points in these edges that are also in the edge $[P_2, P_6 + e_1]$ therefore $[P_2, P_6 + e_1]$ does not intersect $[P_5 + e_1, P_0 + e_1]$ nor $[P_5 + e_1, P_4 + e_1]$. Finally, edges $[P_2, P_6 + e_1]$ and $[P_5 + e_1, P_6 + e_1]$ intersect in $P_6 + e_1$. Moreover $y_5 < y_2$ by lemma 3.3.3, then $P_5 + e_1$ is not contained on the segment $[P_2, P_6 + e_1]$, hence, $P_6 + e_1$ is the only intersection point of $[P_2, P_6 + e_1]$ and $[P_5 + e_1, P_6 + e_1]$. We deduce that $[P_2, P_6 + e_1]$ does not intersect the hexagon in D_0 or in $D_0 + e_1$ and then any hexagon in $D_0 + ae_1 + be_2$ for all $a, b \in \mathbb{Z}$.

6. $[P_3, P_5 + e_1]$: The lift of $[p_3, p_5]$ is $[P_3, P_5 + e_1]$. Every point $Q := (x, y)$ contained in $[P_3, P_5 + e_1]$ verifies, by lemma 3.3.3 and equations 3.3.2 that $x_3 < x < x_5 + 1$ and $y_3 < y < y_5$. Every point $R := (r_1, r_2)$

different of P_5+e_1 contained in the lifts of $[p_i, p_j]$ on D_0+e_1 verifies that $x_5+1 < x$, then any point in the hexagon in D_0+e_1 can be at the same time in $[P_3, P_5+e_1]$ and in edges in D_0+e_1, then $[P_3, P_5+e_1]$ does not intersect the any lift of $[p_i, p_j]$ in D_0+e_1. Now, every point $S := (s_1, s_2)$ different of P_3 in $[P_0, P_1]$, $[P_6, P_1]$, $[P_0, P_6]$, $[P_6, P_5]$, $[P_0, P_5]$, $[P_5, P_4]$, $[P_0, P_4]$, $[P_4, P_3]$, and $[P_0, P_3]$ is, by lemma 3.3.3, such that $s_1 < x_3$, then $[P_3, P_5+e_1]$ can not intersect any of these edges. For the points $T := (t_1, t_2)$ in $[P_0, P_2]$ and $[P_1, P_2]$, we know by lemma 3.3.3 that $t_2 > y_5$, thus $[P_3, P_5+e_1]$ can not intersect any of these edges. Finally, edges $[P_3, P_5+e_1]$ and $[P_3, P_2]$ have P_3 as common point. Since $y_2 > y_5$, then $P_2 \notin [P_3, P_5+e_1]$ and then P_3 is their only common point. We deduce that $[P_3, P_5+e_1]$ does not intersect any lift of $[p_i, p_j]$ in D_0. We conclude that this edge does not intersect any lift of $[p_i, p_j]$ in $D_0+ae_1+be_2$ for all $a, b \in \mathbb{Z}$.

7. $[P_1, P_4+e_1+e_2]$: The lift of $[p_1, p_4]$ is $[P_1, P_4+e_1+e_2]$. This edge can be contained in D_0, D_0+e_1 , D_0+e_2 and $D_0+e_1+e_2$. Then we verify that it does not intersect the hexagons in these domains. Every point $Q = (x, y)$ contained in $[P_1, P_4+e_1+e_2]$ verifies that $x_1 < x < x_4+1$ and $y_1 < y < y_4+1$. Now, every point $R := (R_1, R_2)$ different of $P_4+e_1+e_2$ and P_4+e_2, in the hexagons contained in $D_0+e_1+e_2$ and in D_0+e_2, verifies, by lemma 3.3.3, that $r_2 > y+4+1$. Since $x_4 < x_1$, the point P_4+e_2 is completely in the left of $[P_1, P_4+e_1+e_2]$. We have also that every point $S := (s_1, s_2)$ different to P_1 and P_1+e_1, contained in the hexagon in D_0 or in D_0+e_1 verifies, by the same lemma, that $s_2 < y_1$. Since $x_4+1 < x_1+1$ by equations 3.3.2, the point P_1+e_1 is completely at the right of $[P_1, P_4+e_1+e_2]$. These inequalities imply that any point in the hexagon in D_0, in D_0+e_2, in D_0+e_1 or in $D_0+e_1+e_2$, outside P_1 and $P_4+e_1+e_2$, could be also contained in $[P_1, P_4+e_1+e_2]$, then $[P_1, P_4+e_1+e_2]$ does not intersect these hexagons.

8. $[P_1, P_5+e_1+e_2]$: The lift of $[p_1, p_5]$ est $[P_1, P_5+e_1+e_2]$. This edge could be contained in D_0, D_0+e_1, D_0+e_2 or $D_0+e_1+e_2$. All the points $Q = (x, y)$ contained in $[P_1, P_5+e_1+e_2]$ verify that $x_1 < x < x_5+1$ and $y_1 < y < y_5+1$. Now, every point $R := (r_1, r_2)$ different to $P+5+e_1+e_2$ or P_5+e_1 contained in the hexagon in $D_0+e_1+e_2$ or in D_0+e_2 verifies that $r_1 > x_5+1$, by lemma 3.3.3, we also know that $y_5 < y_1$, which means that the point P_5+e_1 is completely under the segment $[P_1, P_5+e_1+e_2]$. Moreover, every point $S := (s_1, s_2)$ different to P_1 contained in the hexagon in D_0, verifies that $s_2 < y_1$. This and the previous argument means that $[P_1, P_5+e_1+e_2]$ can not intersect the hexagons in D_0, D_0+e_1 or in $D_0+e_1+e_2$ outside the points P_1 and $P_5+e_1+e_2$. Every point $T := (t_1, t_2)$ different to P_1+e_2 contained in

$[P_1+e_2, P_0+e_2]$, $[P_1+e_2, P_6+e_2]$, $[P_6+e_2, P_0+e_2]$, $[P_6+e_2, P_5+e_2]$, $[P_5+e_2, P_0+e_2]$, $[P_5+e_2, P_4+e_2]$, $[P_4+e_2, P_0+e_2]$, $[P_4+e_2, P_3+e_2]$ and $[P_3+e_2, P_0+e_2]$ verify that $t_1 < x_1$ and since $y_1+1 > y_5+1$ by lemma 3.3.3, we have that the point P_1+e_2 is completely over the segment $[P_1, P_5+e_1+e_2]$. We conclude that $[P_1+e_2+e_2]$ does not intersect any of these edges. Moreover, every point $U := (u_1, u_2)$ contained in $[P_2+e_2, P_1+e_1]$ and $[P_2+e_2, P_0+e_0]$ verifies that $u_2 > y_5+1$ by lemma 3.3.3, then $[P_1, P_5+e_1+e_2]$ does not intersect these edges. Finally, in the sixth case we showed that $[P_3, P_5+e_1]$ does not intersect any hexagon, then $[P_3+e_1, P_5+e_1+e_2]$ does not intersect any hexagon, in particular the hexagons in $D_0+e_1+e_2$ and in D_0+e_2. Now, we know that $P_5+e_1+e_2 \in [P_3, P_5+e_1+e_2]$ and by lemma 3.3.3, every point $V := (v_1, v_2) \in [P_3+e_2, P_5+e_1+e_2]$ different to P_3+e_3, verifies that $u_2 > y_3+1$. Since $y_1 < y_3+1$ by equations 3.3.2, P_1 is not contained in this segment, in fact, the last inequality shows that P_1 is completely under the segment $[P_3+e_2, P_5+e_1+e_2]$, and since, by lemma 3.3.3, every point $Q = (x, y) \in [P_1, P_5+e_1+e_2]$ verifies that $x_3 < x < x_5+1$, then the segment $[P_1, P_5+e_1+e_2]$ is completely under the segment $[P_3+e_2, P_5+e_1+e_2]$ outside their intersection point $P_5+e_1+e_2$. Similarly,every point $W := (w_1, w_2)$ different to P_3+e_2, contained in $[P_3+e_2, P_2+e_2]$ verifies that $x_3 < w_1 < x_5+1$ and $y_3+1 < w_2$. and since $y_2+1 > y_5+1$ then P_2+e_2 is not contained in $[P_3+e_2, P_5+e_1+e_2]$ and is completely over this segment. We deduce that $[P_3+e_2, P_2+e_2]$ is completely over the segment $[P_3+e_2, P_5+e_1+e_2]$ outside their intersection point P_3+e_2. In we suppose then that $[P_1, P_5+e_1+e_2]$ intersects the segment $[P_3+e_2, P_2+e_2]$, then there is a point I different to $P_5+e_1+e_2$ in $[P_1, P_5+e_1+e_2]$ that intersects $[P_3+e_2, P_5+e_1+e_2]$, which is a contradiction because $P_5+e_1+e_2$ is their only intersection point. Then, $[P_1, P_5+e_1+e_3]$ does not intersect any edge of the hexagon in D_0+e_2, and in consequence, any hexagon in $D_0+ae_1+be_2$ for all $a, b \in \mathbb{Z}$.

9. $[P_2, P_4+e_1+e_2]$: The lift of $[p_2, p_4]$ is $[P_2, P_4+e_1+e_2]$. Every point $Q := (x, y)$ in $[P_2, P_4+e_1+e_2]$ verifies that $x_2 < x < x_4+1$ and $y_2 < y < y_4+1$. Now, every point $R := (r_1, r_2)$ different to P_2 and P_2+e_2, in the hexagon D_0 and in D_0+e_2 verifies that $r_1 < x_2$, and also we know by lemma 3.3.3 that $y_2+1 > y_4+1$ then $[P_2, P_4+e_1+e_2]$ does not intersect any point contained in the hexagons in D_0 or in D_0+e_2 outside P_2. By lemma 3.3.3, we know also that every point $S := (s_1, s_2)$ different to $P_4+e_1+e_2$ in the hexagon in $D_0+e_1+e_2$ verifies that $s_2 > y_4+1$, then $[P_2, P_4+e_1+e_2]$ does not intersect the hexagon in $D_0+e_1+e_2$. We have that every point $T := (t_1, t_2)$, different of P_4+e_1, contained in $[P_6+e_1, P_1+e_1]$, $[P_6+e_1, P_0+e_1]$, $[P_1+e_1, P_2+e_1]$, $[P_1+e_1, P_0+e_1]$, $[P_2+e_1, P_3+e_1]$, $[P_2+e_1, P_0+e_1]$,

$[P_3 + e_1, P_4 + e_1]$, $[P_3 + e_1, P_0 + e_1]$ and $[P_4 + e_1, P_0 + e_1]$, verifies that $t_1 > x_4+1$ and since $y_4 < y_1$, P_4+e_1 is not contained in $P_2, P_4+e_1+e_2]$. We deduce that $[P_2, p_4 + e_1 + e_2]$ does not intersect any of these edges. We have that any point $U := (u_1, u_2)$ in $[P_3 + e_1, P_0]$, $[P_3 + e_1, P_4 + e_1]$ verifies that $u_2 < y_2$, then $[P_2, P_4 + e_1 + e_2]$ does not intersect these edges neither. Finally, since $y_4 + 1 > y_6$ and $y_3 < y_2$, the segments $[P_2, P_4+e_1+e_2]$ and $[P_3+e_1, P_6+e_1]$ are not contained in the segment $[P_2, P_6 + e_1]$ as every point $V := (v_1, v_2)$ in this segment verifies that $y_2 < v_2 < y_6+1$. Since $x_3+1 > x_2$ and $x_4+1 < x_6+1$, we deduce that the segments $[P_2, P_4 + e_1 + e_2]$ and $[P_3 + e_1, P_6 + e_1]$ are completely over and under the segment $[P_2, P_6 + e_1]$ respectively, outside their intersection point (P_2 and $P_6 + e_1$ respectively). If $[P_2, P_4 + e_1 + e_2]$ intersects $[P_3 + e_1, P_6 + e_1]$, then there is, by continuity, a point $I \in [P_3 + e_1, P_6 + e_1]$ different to $P_6 + e_1$ and $P_3 + e_1$, where $[P_2, P_6 + e_1]$ and $[P_2, P_4 + e_1 + e_2]$ intersects, but since P_2 is in the intersection of these two segments and it is the only intersection point, then we have a contradiction. We conclude that $[P_2, P_4 + e_1 + e_2]$ does not intersects any point in the hexagon in $D_0 + e_1$, therefore any hexagon in $D_0 + ae_1 + be_2$ for all $a, b \in \mathbb{Z}$.

We have proved that the 9 edges that do not conform the hexagon in D_0 do not intersect this hexagon. That means that around P_0 there is a triangulation consisting of a cycle of 6 triangles since all properties that are true for D_0 are also true for D_i. By lemma 3.3.5, we conclude that around any point P_i, there is a triangulation consisting in a cycle of 6 triangles. Hence, the lifts of the edges $[p_k, p_l]$ do not intersect. We deduce that we are in the case $C_{I,3}$.

$\Rightarrow$) Reciprocally, suppose that we are in a configuration type $C_{I,3}$, since $P_3 \in Q_2$, we know in which quadrant is contained every other point P_i. We will prove that equations $(1) - (7)$ and $(1') - (7')$ are verified.

For that, note that $[P_i, P_{i+1}]$ and $[P_0, P_i]$ are lifts of $[p_i, p_{i+1}]$ and $[p_0, p_i]$ for all $i = 1, \ldots, 6$ for every configuration, this is because the lift of Starp_0 is StarP_0 and the shortest path property implies that StarP_0 is contained in D_0 (see figure 3.8). We will now verify which are the lifts of the other 9 edges.

1. Lift of $[p_1, p_3]$: Since $P_1 \in Q_1$ and $P_3 \in Q_3$, then, by the shortest path property, the possible lifts for $[p_1, p_3]$ are $[P_1, P_3]$ and $[P_1, P_3 + e_2]$.

Case 1: If $[P_1, P_3]$ is the lift of $[p_1, p_3]$, then, since $[P_0, P_1]$ and $[P_0, P_3]$ are lifts of $[p_0, p_1]$ and $[p_0, p_3]$, then, by lemma 3.2.4 $[P_0, P_1, P_3]$ is a lift of a triangle, that is, $[p_0, p_1, p_3] \in \mathcal{T}$, which means that $(013) \in LI$, the list of proposition 3.1.1, which is a contradiction.

Case 2: The contradiction in case 1 implies that $[P_1, P_3 + e_2]$ is the lift of $[p_1, p_3]$.

2. Lift of $[p_6, p_4]$: Since $P_6 \in Q_4$ and $P_4 \in Q_3$, then, by the shortest path property, the possible lifts for $[p_6, p_4]$ are $[P_6, P_4]$ and $[P_6, P_4 + e_2]$.

Case 1: If $[P_6, P_4]$ is the lift of $[p_6, p_4]$, then, since $[P_0, P_6]$ and $[P_0, P_4]$ are lifts of $[p_0, p_6]$ and $[p_0, p_4]$, then, by lemma 3.2.4 $[P_0, P_6, P_3]$ is a lift of a triangle, the triangle $[p_0, p_6, p_3] \in \mathcal{T}$, which means that $(064) \in LI$, which is a contradiction.

Case 2: The contradiction in case 1 implies that $[P_6, P_4 + e_2]$ is the lift of $[p_6, p_4]$.

3. Lift of $[p_3, p_5]$: Since $P_3 \in Q_2$ and $P_5 \in Q_3$, the possible lifts for $[p_3, p_5]$ are $[P_3, P_5]$ and $[P_3, P_5 + e_1]$.

Case 1: If $[P_3, P_5]$ is the lift of $[p_3, p_5]$, then, since $[P_0, P_3]$ and $[P_0, P_5]$ are lifts of $[p_0, p_3]$ and $[p_0, p_5]$, then by lemma 3.2.4 $[P_0, P_3, P_5]$ is a lift of a triangle, which means that $(035) \in LI$, we have then a contradiction.

Case 2: The contradiction in case 1 implies that $[P_3, P_5 + e_1]$ is the lift of $[p_3, p_5]$.

4. Lift of $[p_2, p_6]$: Since $P_2 \in Q_1$ and $P_3 \in Q_4$, the possible lifts for $[p_2, p_6]$ are $[P_2, P_6]$ and $[P_2, P_6 + e_1]$.

Case 1: If $[P_2, P_6]$ is the lift of $[p_2, p_6]$, then, since $[P_0, P_2]$ and $[P_0, P_6]$ are lifts of $[p_0, p_2]$ and $[p_0, p_6]$, then by lemma 3.2.4 $[P_0, P_3, P_5]$ is a lift of a triangle which means that $(035) \in LI$, which is a contradiction.

Case 2: The contradiction in case 1 implies that $[P_2, P_6 + e_1]$ is the lift of $[p_2, p_6]$.

5. Lift of $[p_3, p_6]$: The possible lifts for $[p_3, p_6]$ are $[P_6, P_3]$, $[P_6, P_3 - e_1]$, $[P_6, P_3 + e_2]$ or $[P_6, P_3 - e_1 + e_2]$.

Case 1: If $[P_3, P_6]$ is the lift of $[p_6, p_3]$, since $[P_0, P_3]$ and $[P_0, P_6]$ are lifts of $[p_0, p_3]$ and $[p_0, p_6]$, then by lemma 3.2.4, $[P_0, P_3, P_6]$ is a lift of a triangle, hence $(063) \in LI$ which is a contradiction.

Case 2: If $[P_6, P_3 - e_1]$ is a lift of $[p_6, p_3]$, then, since $[P_5, P_6]$, is a lift of $[p_5, p_6]$ and as we saw previously, $[P_3, P_5 + e_1]$ is a lift of $[p_3, p_5]$, that is $[P_3 - e_1, P_5]$ is a lift of $[p_3, p_5]$, then by lemma 3.2.4, $[P_3 - e_1, P_6 + e_1, P_5]$ is the lift of a triangle, then $(365) \in LI$ which is a contradiction.

Case 3: If $[P_6, P_3 - e_1 + e_2]$ is a lift of $[p_3, p_6]$. We saw that $[P_6, P_4 + e_1]$ is a lift of $[p_6, p_4]$. Since $(634) \in LI$ and $[P_6, P_3 - e_1 + e_2]$ and $[P_6, P_4 + e_1]$ are lifts of $[p_3, p_6]$ and $[p_6, p_4]$, then $[P_3, P_4 + e_1]$ is a lift of $[p_3, p_4]$ which is a contradiction.

Case 4: Since any of the other cases are possible, then the lift of $[p_3, p_6]$ is $[P_6, P_3 + e_2]$.

6. Lift of $[p_1, p_5]$: The possible lift for $[p_1, p_5]$ are $[P_1, P_5]$, $[P_1, P_5 + e_1]$, $[P_1, P_5 + e_2]$ or $[P_1, P_5 + e_1 + e_2]$

Case 1: If $[P_1, P_5]$ is a lift of $[p_1, p_5]$, since $[P_0, P_1]$ and $[P_0, P_5]$ are lifts of $[p_0, p_1]$ and $[p_0, p_5]$ respectively, then $[P_0, P_1, P_5]$ is the lift of a triangle and this means that $(015) \in LI$ which is a contradiction.

Case 2: We know that $(154) \in LI$ and that $[P_4, P_5]$ is a lift of $[p_4, p_5]$. If we suppose that $[P_1, P_5 + e_2]$ is a lift of $[p_1, p_5]$, then $[P_1, P_5 + e_2, P_4 + e_2]$ is a lift of a triangle. This means that $[P_1, P_4 + e_2]$ must be a lift of $[p_1, p_4]$. If this is the case, since $[P_1, P_3 + e_2]$ is a lift of $[p_1, p_3]$ as seen previously and $[P_3 + e_2, P_4 + e_2]$ is a lift of $[p_3, p_4]$, then $[P_1, P_3 + e_2, P_4 + e_2]$ is a lift of a triangle by lemma 3.2.4, but $(134) \notin LI$, then $[P_1, P_4 + e_2]$ can not be a lift of $[p_1, p_4]$, therefore $[P_1, P_5 + e_2]$ is not a lift of $[p_1, p_5]$.

Case 3: Suppose that $[P_1, P_5 + e_1]$ is a lift of $[p_1, p_5]$, since $(153) \in LI$ and $[P_3, P_5 + e_1]$ is a lift of $[p_3, p_5]$, then necessarily we have that $[P_1, P_3]$ is a lift of $[p_1, p_3]$, which is a contradiction as we know that $[P_1, P_3 + e_2]$ is a lift of $[p_1, p_3]$

Case 4: Because the cases 1-3 are not possible, then the lift pf $[p_1, p_5]$ is $[P_1, P_5 + e_1 + e_2]$.

7. Lift of $[p_1, p_4]$: The possible lifts for $[p_1, p_4]$ are $[P_1, P_4]$, $[P_1, P_4 + e_1]$, $[P_1, P_4 + e_2]$ or $[P_1, P_4 + e_1 + e_2]$.

Case 1: If $[P_1, P_4]$ is a lift of $[p_1, p_4]$, since $[P_0, P_1]$ and $[P_0, P_4]$ are lifts of $[p_0, p_1]$ and $[p_0, p_4]$, then $[P_1, P_0, P_4]$ is a lift of a triangle by lemma 3.2.4, then $(104) \in Li$ which is a contradiction.

Case 2: $[P_1, P_4 + e_2]$ can not be a lift of $[p_1, p_4]$ as we showed in case 2 form lift of $[p_1, p_5]$.

Case 3: Suppose that $[P_1, P_4 + e_1]$ is a lift of $[p_1, p_4]$. We know that $[P_5 + e_1, P_4 + e_1]$ is a lift of $[p_5, p_4]$ and that $(154) \in LI$, then necessarily $[P_1, P_5 + e_1]$ is a lift of $[p_1, p_5]$, which is a contradiction.

Case 4: Since any of the previous cases are possible, we conclude that $[P_1, P_4 + e_1 + e_2]$ is a lift of $[p_1, p_4]$.

8. Lift of $[p_2, p_4]$: The possible lifts for $[p_2, p_4]$ are $[P_2, P_4]$, $[P_2, P_4 + e_1]$, $[P_2, P_4 + e_2]$, $[P_2, P_4 + e_1 + e_2]$.

Case 1: If $[P_2, P_4]$ is a lift of $[p_2, p_4]$, since $[P_0, P_2]$, $[P_0, P_4]$ are lifts of $[p_0, p_2]$ and $[p_0, p_4]$ respectively, by lemma 3.2.4 we have that $[P_0, P_2, P_4]$ must be a lift of a triangle of $\mathcal{T}$, that means that $(024) \in LI$ which is a contradiction.

Case 2: Suppose that $[P_2, P_4 + e_2]$ is a lift of $[p_2, p_4]$. Since $[P_2, P_1]$ is a lift of $[p_2, p_1]$ and we know that $(241) \in LI$, then necessary $[P_1, P_4 + e_2]$ is a lift of $[p_1, p_4]$, which is a contradiction.

Case 3: Suppose that $[P_2, P_4 + e_1]$ is a lift of $[p_2, p_4]$. Since $[P_2, P_1]$ is a lift of $[p_2, p_1]$ and we know that $(241) \in LI$, then necessary $[P_1, P_4 + e_1]$ is a lift of $[p_1, p_4]$, which is a contradiction.

Case 4: The fact that the other three cases are not possible means that $[P_2, P_4 + e_1 + e_2]$ is a lift of $[p_2, p_4]$.

9. Lift of $[p_2, p_5]$: The possible lifts for $[p_2, p_5]$ are $[P_2, P_5]$, $[P_2, P_5 + e_1]$, $[P_2, P_5 + e_2]$ or $[P_2, P_5 + e_1 + e_2]$.

Case 1: If $[P_2, P_5]$ is a lift of $[p_2, p_5]$, since $[P_0, P_2]$ and $[P_0, P_5]$ are lifts of $[p_0, p_2]$ and $[p_0, p_5]$, then $[P_0, P_2, P_5]$ is a lift of a triangle by lemma 3.2.4, then $(025) \in LI$ which is a contradiction.

Case 2: Suppose that $[P_2, P_5 + e_2]$ is a lift of $[p_2, p_5]$. Since $[P_2, P_3]$ is a lift of $[p_2, p_3]$ and $(253) \in LI$, then necessarily $[P_2, P_5 + e_2, P_3]$ is a lift of a triangle, hence $[P_3, P_5 + e_2]$ is a lift of $[p_3, p_5]$, but we know that a lift of $[p_3, p_5]$ is $[P_3, P_5 + e_1]$, then $[P_2, P_5 + e_2]$ can not be a lift of $[p_2, p_5]$.

Case 3: Suppose that $[P_2, P_5 + e_1 + e_2]$ is a lift of $[p_2, p_5]$. Since $[P_2, P_3]$ is a lift of $[p_2, p_3]$ and $(235) \in LI$, then $[P_2, P_5 + e_1 + e_2, P_3]$ is a lift of a triangle, therefore $[P_3, P_5 + e_1 + e_2]$ is a lift of $[p_3, p_5]$ which is a contradiction.

Case 4: Since any of the previous cases are possible, then $[P_2, P_5 + e_1]$ is a lift of $[p_2, p_5]$.

Once we know all the lifts of the edges $[p_i, p_j]$, the equations are deduced since, by lemma 3.2.2 we have that if $[P_i + ae_1 + be_2, P_j + ce_1 + de_2]$ is a lift of $[p_i, p_j]$, then $|x_i + a - (x_j + c)| < \frac{1}{2}$ and $|y_i + b - (y_j + d)| < \frac{1}{2}$.

□

Remark 3.3.8. *We have studied properties of configuration $C_{I,3}$, that is, the second quadrant Q_2 of D_0 contains only one point, the point P_3. But relabeling, the same results are true for configurations of type $C_{I,i}$, that is, configurations where the only point in Q_2 of D_0 is the point P_i for $i = 0, \ldots, 6$.*

Similarly, if we relabel, all the properties are true for configurations of type $C_{II,i}$, consisting of configurations where the quadrant Q_1 in D_0 contains only the point P_i.

Theorem 3.3.9. *The equations given in 3.3.2 describe two simplexes of dimension 6, so the configuration space for configuration type $C_{I,3}$ can be denoted by $\Delta_x \times \Delta_y$.*

Proof. First of all, note that the coordinates x_i verify $-\frac{1}{2} < x_i < \frac{1}{2}$ and y_i verify also $-\frac{1}{2} < y_i < \frac{1}{2}$. This implies that the space described by the vertices P_i, for $i = 0, \dots, 6$ is bounded. Furthermore, we have seven equations in $\mathbb{R}^6$ for the first coordinate (and seven equations in $\mathbb{R}^6$ for the second coordinate) that define the intersection of seven half-spaces. We prove now that this intersection is non empty. Take $P_0 = (0,0), P_1 = \left(\frac{2}{7}, \frac{3}{7}\right), P_2 = \left(\frac{3}{7}, \frac{1}{7}\right), P_3 = \left(\frac{1}{7}, -\frac{2}{7}\right), P_4 = \left(-\frac{2}{7}, -\frac{3}{7}\right), P_5 = \left(-\frac{3}{7}, -\frac{1}{7}\right)$ and $P_6 = \left(-\frac{1}{7}, \frac{2}{7}\right)$. We show that these points verify the equations given in 3.3.2. Indeed, we have:

(1). $x_2 - x_0 = \frac{3}{7} < \frac{1}{2}$

(2). $x_6 - x_2 = -\frac{4}{7} < -\frac{1}{2}$

(3). $x_1 - x_6 = \frac{3}{7} < \frac{1}{2}$

(4). $x_4 - x_1 = -\frac{4}{7} < -\frac{1}{2}$

(5). $x_3 - x_4 = \frac{3}{7} < \frac{1}{2}$

(6). $x_5 - x_3 = -\frac{4}{7} < -\frac{1}{2}$

(7). $x_0 - x_5 = \frac{3}{7} < \frac{1}{2}$

and

(1'). $y_1 - y_0 = \frac{3}{7} < \frac{1}{2}$

(2'). $y_5 - y_1 = -\frac{4}{7} < -\frac{1}{2}$

(3'). $y_6 - y_5 = \frac{3}{7} < \frac{1}{2}$

(4'). $y_3 - y_6 = -\frac{4}{7} < -\frac{1}{2}$

(5'). $y_2 - y_3 = \frac{3}{7} < \frac{1}{2}$

(6'). $y_4 - y_2 = -\frac{4}{7} < -\frac{1}{2}$

(7'). $y_0 - y_4 = \frac{3}{7} < \frac{1}{2}$

We have a non empty bounded intersection of seven half-spaces, we deduce then that this intersection is a 6-simplex. In general, the equations given in 3.3.2 define a product of two six-simplex, denoted by Δ_x for the simplex defined by equations $(1) - (7)$ and Δ_y for the simplex defined by equations $(1') - (7')$. In particular, all the triangulations of $C_{I,3}$ are contained in a space of dimension 12. □

By remark 3.3.8, the other 11 cases presented have similar equations as configuration $C_{I,3}$, we deduce that the equations associated to these configurations define a product of 6-simplexes. We have then the following corollary:

Corollary 3.3.10. *The configuration space $GE(M, \mathbb{T}^2)$ is given by*

$$GE(M, \mathbb{T}^2) = \bigcup_{i=1}^{12} (\Delta_x \times \Delta_y)_i,$$

where $\Delta_x \times \Delta_y$ denote the product of two (different) 6-simplex, Δ_x and Δ_y.

Lemma 3.3.11. *The vertices of the closure of Δ_x defined by equations 3.3.2 are given by:*

- $v_1 = \left(0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
- $v_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$
- $v_3 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{2}, 0\right)$
- $v_4 = \left(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0\right)$
- $v_5 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0\right)$
- $v_6 = \left(0, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0\right)$
- $v_7 = \left(0, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

On the other hand, the vertices of the closure of Δ_y defined by equations 3.3.2 are given by:

- $w_1 = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right)$
- $w_2 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}\right)$
- $w_3 = \left(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}\right)$
- $w_4 = \left(\frac{1}{2}, 0, 0, -\frac{1}{2}, 0, \frac{1}{2}\right)$
- $w_5 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right)$
- $w_6 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$
- $w_7 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}\right)$

Proof. Since the inequalities in 3.3.2 are strict, the two 6-simples Δ_x and Δ_y are open. The closure of them are defined by the equations with non strict inequalities, that is:

(i). $x_2 - x_0 \leq \frac{1}{2}$

(ii). $x_6 - x_2 \leq -\frac{1}{2}$

(iii). $x_1 - x_6 \leq \frac{1}{2}$

(iv). $x_4 - x_1 \leq -\frac{1}{2}$

(v). $x_3 - x_4 \leq \frac{1}{2}$

(vi). $x_5 - x_3 \leq -\frac{1}{2}$

(vii). $x_0 - x_5 \leq \frac{1}{2}$

and

(i'). $y_1 - y_0 \leq \frac{1}{2}$

(ii'). $y_5 - y_1 \leq -\frac{1}{2}$

(iii'). $y_6 - y_5 \leq \frac{1}{2}$

(iv'). $y_3 - y_6 \leq -\frac{1}{2}$

(v'). $y_2 - y_3 \leq \frac{1}{2}$

(vi'). $y_4 - y_2 \leq -\frac{1}{2}$

(vii'). $y_0 - y_4 \leq \frac{1}{2}$

These equations imply:

(1-1). $0 \leq x_1 \leq \frac{1}{2}$

(2-1). $0 \leq x_2 \leq \frac{1}{2}$

(3-1). $0 \leq x_3 \leq \frac{1}{2}$

(4-1). $-\frac{1}{2} \leq x_4 \leq 0$

(5-1). $-\frac{1}{2} \leq x_5 \leq 0$

(6-1). $-\frac{1}{2} \leq x_6 \leq 0$

and

(1-2). $0 \leq y_1 \leq \frac{1}{2}$ (3-2). $-\frac{1}{2} \leq y_3 \leq 0$ (5-2). $-\frac{1}{2} \leq y_5 \leq 0$

(2-2). $0 \leq y_2 \leq \frac{1}{2}$ (4-2). $-\frac{1}{2} \leq y_4 \leq 0$ (6-2). $0 \leq y_6 \leq \frac{1}{2}$

We will now find the vertices of the two 6-simplex Δ_x and Δ_y. Every vertex of Δ_x has the form $v_k = (x_1, x_2, x_3, x_5, x_5, x_6)$ for $k = 1, \ldots, 7$, where $x_i \in \left\{0, \frac{1}{2}\right\}$ for $i = 1, 2, 3$ and $x_j \in \left\{-\frac{1}{2}, 0\right\}$ for $j = 1, 2, 3$. Indeed, if x_i or x_j do not take these values, then they are at the interior of Δ_x (or at the exterior) thus v_k could not be a vertex. Similarly, every vertex v_l of the closure of Δ_y has the form $v_l = (y_1, y_2, y_3, y_4, y_5, y_6)$, where $y_1, y_2, y_6 \in \left\{0, \frac{1}{2}\right\}$ and $y_1, y_4, y_5 \in \left\{-\frac{1}{2}, 0\right\}$. By equations $(i) - (vii)$ and $(1-1) - (6-1)$, we have the following cases:

1. If $x_1 = 0$, then $x_4 = -\frac{1}{2}$, $x_3 = 0$ and $x_5 = -\frac{1}{2}$.
2. If $x_1 = \frac{1}{2}$, then $x_6 = 0$ and $x_2 = \frac{1}{2}$.
3. If $x_2 = 0$, then $x_6 = -\frac{1}{2}$, $x_1 = 0$, $x_4 = -\frac{1}{2}$, $x_3 = 0$ and $x_5 = -\frac{1}{2}$.
4. If $x_2 = \frac{1}{2}$, then there is no other point than can be fixed.
5. If $x_3 = 0$, then $x_5 = -\frac{1}{2}$.
6. If $x_3 = \frac{1}{2}$, then $x_1 = \frac{1}{2}$, $x_4 = 0$, $x_6 = 0$ and $x_2 = \frac{1}{2}$.
7. If $x_4 = 0$, then $x_1 = \frac{1}{2}$, $x_6 = 0$ and $x_2 = \frac{1}{2}$.
8. If $x_4 = -\frac{1}{2}$, then $x_3 = 0$ and $x_5 = -\frac{1}{2}$.
9. If $x_5 = 0$, then $x_3 = \frac{1}{2}$, $x_4 = 0$, $x_1 = \frac{1}{2}$, $x_6 = 0$ and $x_2 = \frac{1}{2}$.
10. If $x_5 = -\frac{1}{2}$, then there is no other point that can be fixed.
11. If $x_6 = 0$, then $x_2 = \frac{1}{2}$.
12. If $x_6 = -\frac{1}{2}$, then $x_1 = 0$, $x_4 = -\frac{1}{2}$, $x_3 = 0$ and $x_5 = -\frac{1}{2}$.

In the third and ninth cases, we have all points fixed, that is, we have the vertices $v_1 = \left(0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ and $v_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. This implies that any other vertex has to be such that $x_2 = \frac{1}{2}$ and $x_5 = -\frac{1}{2}$.

If $x_1 = \frac{1}{2}$, by the second case and the previous argument, we have a vertex of the form $v_k = \left(\frac{1}{2}, \frac{1}{2}, \cdot, \cdot, -\frac{1}{2}, 0\right)$. We can have $x_3 = 0$ because by the fifth case $x_5 = -\frac{1}{2}$ so there is no contradiction, so that the vertex takes the form $v_k = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdot, -\frac{1}{2}, 0\right)$. Seventh case implies no contradiction so we can have $x_4 = 0$. In this case, the vertex becomes $v_3 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{2}, 0\right)$. Also, eighth case implies no contradiction and in this case, the vertex becomes $v_4 = \left(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0\right)$. On the other hand, if $x_3 = \frac{1}{2}$, by the sixth case, we have $x_4 = 0$ and we have no contradictions with the other points. In this case, the vertex takes the form $v_5 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0\right)$. We have studied all cases where $x_1 = \frac{1}{2}$.

If $x_1 = 0$, by the first case and a previous argument, the vertices will take the form $v_k = \left(0, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, \cdot\right)$. By the eleventh case, we can have $x_6 = 0$ with no contradiction, so than the vertex becomes $v_6 = \left(0, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0\right)$. By the twelfth case, we can have $x_6 = \frac{1}{2}$ which do not lead to any contradiction, so that the vertex becomes $v_7 = \left(0, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. We have studied all the cases where $x_1 = 0$.

These are all vertices since we have studied all possible case for x_1, moreover Δ_x is a 6-simplex so we have just 7 vertices.

We now determine the vertices of Δ_y. By equations $(i') - (vii')$ and $(1-2)$ to $(6-2)$, we have the following cases:

1. If $y_1 = 0$, then $y_5 = -\frac{1}{2}$, $y_6 = 0$, $y_3 = -\frac{1}{2}$, $y_2 = 0$ and $y_4 = -\frac{1}{2}$
2. If $y_1 = \frac{1}{2}$, then there is no other point that can be fixed.
3. If $y_2 = 0$, then $y_4 = -\frac{1}{2}$.
4. If $y_2 = \frac{1}{2}$, then $y_3 = 0$, $y_6 = \frac{1}{2}$, $y_5 = 0$ and $y_1 = \frac{1}{2}$.
5. If $y_3 = 0$, then $y_6 = \frac{1}{2}$, $y_5 = 0$ and $y_1 = \frac{1}{2}$.
6. If $y_3 = -\frac{1}{2}$, then $y_2 = 0$ and $y_4 = -\frac{1}{2}$.
7. If $y_4 = 0$, then $y_2 = 0$, $y_3 = 0$, $y_6 = \frac{1}{2}$, $y_5 = 0$ and $y_1 = \frac{1}{2}$.
8. If $y_4 = -\frac{1}{2}$, then there is no other point that can be fixed.
9. If $y_5 = 0$, then $y_1 = \frac{1}{2}$.
10. If $y_5 = -\frac{1}{2}$, then $y_6 = 0$, $y_3 = -\frac{1}{2}$, $y_2 = 0$ and $y_4 = -\frac{1}{2}$.
11. If $y_6 = 0$, then $y_3 = -\frac{1}{2}$, $y_2 = 0$ and $y_4 = -\frac{1}{2}$.
12. If $y_6 = \frac{1}{2}$, then $y_5 = 0$ and $y_1 = \frac{1}{2}$.

If $y_1 = 0$ the remaining points are fixed as seen in the first case. In this case, the vertex becomes $w_1 = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right)$. The case where $y_4 = 0$ is similar, the vertex becoming $w_2 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}\right)$. This implies that every other vertex must be such that $y_1 = \frac{1}{2}$ and $y_4 = -\frac{1}{2}$, then of the form $w_l = \left(\frac{1}{2}, \cdot, \cdot, -\frac{1}{2}, \cdot, \cdot\right)$.

If $y_2 = \frac{1}{2}$, by the previous argument and the fourth case, we have $w_3 = \left(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}\right)$.

We can have $y_2 = 0$ since it does not imply any contradiction (see third case). In this case, the vertex takes the form $w_l = \left(\frac{1}{2}, 0, \cdot, -\frac{1}{2}, \cdot, \cdot\right)$. If $y_3 = 0$, by the fifth case, we have $w_4 = \left(\frac{1}{2}, 0, 0, -\frac{1}{2}, 0, \frac{1}{2}\right)$. By the sixth case, we can

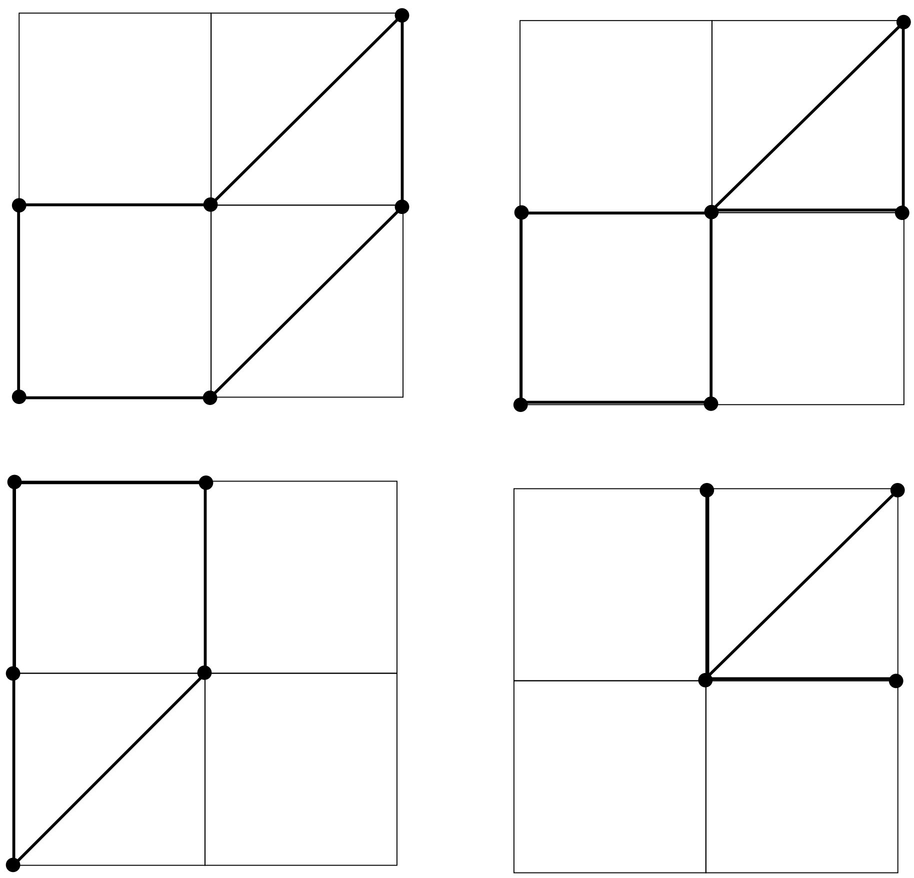

Figure 3.9: Some vertices of $\Delta_x \times \Delta_y$.

have $y_3 = -\frac{1}{2}$, so the vertex becomes $w_l = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, \cdot, \cdot\right)$. We can have $y_5 = -\frac{1}{2}$ without having a contradiction, so that, by the tenth case, we have $w_5 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right)$. We can also have $y_5 = 0$ by the ninth case, then, the vertex takes the form $w_l = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \cdot\right)$. We have no contradiction whether $y_6 = 0$ or $y_6 = \frac{1}{2}$, so that, we have the vertices $w_6 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$ and $w_7 = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}\right)$.

Since we have studied all cases for y_2, then these are the only vertices, in addition to the fact that Δ_y is a 6-simplex (see figure 3.9). □

Example: ***Central configuration.*** Among all the different triangulations of type $C_{I,3}$, there is a special one that we will call *Central configuration*. This configuration consists of seven vertices whose lifts have the following coordinates: $P_0 = (0,0), P_1 = \left(\frac{2}{7}, \frac{3}{7}\right), P_2 = \left(\frac{3}{7}, \frac{1}{7}\right), P_3 = \left(\frac{1}{7}, -\frac{2}{7}\right), P_4 = \left(-\frac{2}{7}, -\frac{3}{7}\right), P_5 = \left(-\frac{3}{7}, -\frac{1}{7}\right)$ and $P_6 = \left(-\frac{1}{7}, \frac{2}{7}\right)$ (see figure 3.10). In theorem

3.3.9 we show that these points verify equations 3.3.2, so these lifts correspond indeed to configuration $C_{I,3}$. We have the following geometrical property of the Central configuration:

Proposition 3.3.12. *In the Central configuration, all the lifts of the faces in* $\mathbb{T}^2$ *are congruent.*

Proof. We calculate the length of the sides of the lifts of all triangles in $\mathcal{T}$. By all the previous results, we know which are lifts of the faces of the triangulation type $C_{I,3}$ in $\mathbb{T}^2$. We denote by $\ell(P_i, P_j)$ the length of the segment $[P_i, P_j]$ and we denote by $[P_i, P_j, P_k] \equiv [P_{i'}, P_{j'}, P_{k'}]$ whenever two triangles are congruent.

1. Take the triangle $[P_0, P_1, P_2]$. We have that $\ell(P_0, P_1) = \frac{\sqrt{13}}{7} := \ell_1$, $\ell(P_0, P_2) = \frac{\sqrt{10}}{7} := \ell_2$ and $\ell(P_1, P_2) = \frac{\sqrt{5}}{7} := \ell_3$.
2. Take the triangle $[P_0, P_2, P_3]$. Then $\ell(P_0, P_2) = \ell_2$, $\ell(P_0, P_3) = \ell_3$ and $\ell(P_2, P_3) = \ell_1$, then $[P_0, P_1, P_2] \equiv [P_0, P_2, P_3]$.
3. Take the triangle $[P_0, P_3, P_4]$. We know that $\ell(P_0, P_3) = \ell_3$, moreover $\ell(P_0, P_4) = \ell_1$ and $\ell(P_4, P_3) = \ell_2$, then $[P_0, P_3, P_4] \equiv [P_0, P_2, P_3]$.
4. Take the triangle $[P_0, P_4, P_5]$. We have $\ell(P_0, P_4) = \ell_1$, on the other hand, $\ell(P_0, P_5) = \ell_2$ and $\ell(P_4, P_5) = \ell_3$, we deduce that $[P_0, P_4, P_5] \equiv [P_0, P_3, P_4]$.
5. Take the triangle $[P_0, P_5, P_6]$. We have $\ell(P_0, P_5) = \ell_2$. We have also $\ell(P_0, P_6) = \ell_3$ and $\ell(P_5, P_6) = \ell_1$. Then $[P_0, P_5, P_6] \equiv [P_0, P_4, P_5]$.
6. Take the triangle $[P_0, P_6, P_1]$. We know that $\ell(P_0, P_6) = \ell_3$ and that $\ell(P_0, P_1) = \ell_1$. On the other hand, $\ell(P_6, P_1) = \ell_2$. We deduce $[P_0, P_6, P_1] \equiv [P_0, P_5, P_6]$.
7. Take the triangle $[P_2, P_6 + e_1, P_5 + e_1]$, we have $P_6 + e_1 = \left(\frac{6}{7}, \frac{2}{7}\right)$ and $P_5 + e_1 = \left(\frac{4}{7}, -\frac{1}{7}\right)$, then $\ell(P_2, P_6 + e_1) = \ell_2$, $\ell(P_2, P_5 + e_1) = \ell_3$ and $\ell(P_6 + e_1, P_5 + e_1) = \ell_1$. Therefore $[P_2, P_6 + e_1, P_5 + e_1] \equiv [P_0, P_6, P_1]$.
8. Take the triangle $[P_2, P_5 + e_1, P_3]$. We know that $\ell(P_2, P_5 + e_1) = \ell_3$ and that $\ell(P_2, P_3) = \ell_1$. We have also $\ell(P_5 + e_1, P_3) = \ell_2$. Then $[P_2, P_5 + e_1, P_3] \equiv [P_2, P_6 + e_1, P_5 + e_1]$.
9. Take the triangle $[P_3, P_5 + e_1, P_1 - e_2]$. We have $P_1 - e_2 = \left(\frac{2}{7}, -\frac{4}{7}\right)$. We know that $\ell(P_3, P_5 + e_1) = \ell_2$. Now, $\ell(P_3, P_1 - e_2) = \ell_3$ and $\ell(P_1 - e_2, P_5 + e_1) = \ell_1$, then $[P_3, P_5 + e_1, P_1 - e_2] \equiv [P_2, P_5 + e_1, P_3]$.
10. Take the triangle $[P_1 - e_2, P_4 + e_1, P_5 + e_1]$. We have $P_4 + e_1 = \left(\frac{5}{7}, -\frac{3}{7}\right)$. We know that $\ell(P_1 - e_2, P_5 + e_1) = \ell_1$. Since translations are isometries, we know also that $\ell(P_4 + e_1, P_5 + e_1) = \ell_3$. Moreover, $\ell(P_1 - e_2, P_4 +$

$e_1) = \ell_2$. We have then that $[P_1 - e_2, P_4 + e_1, P_5 + e_1] \equiv [P_3, P_5 + e_1, P_1 - e_2]$.

11. Take the triangle $[P_1 - e_2, P_4 + e_1, P_2 - e_2]$. We have $P_2 - e_2 = \left(\frac{3}{7}, -\frac{6}{7}\right)$. We know that $\ell(P_1 - e_2, P_4 + e_1) = \ell_2$ and that $\ell(P_1 - e_2, P_2 - e_2) = \ell_3$. On the other hand, we have $\ell(P_4 + e_1, P_2 - e_2) = \ell_1$. Then $[P_1 - e_2, P_4 + e_1, P_2 - e_2] \equiv [P_1 - e_2, P_4 + e_1, P_5 + e_1]$.

12. Take the triangle $[P_2 - e_2, P_4 + e_1, P_6 + e_1 - e_2]$. We have $P_6 + e_1 - e_2 = \left(\frac{6}{7}, -\frac{5}{7}\right)$. We know that $[P_4 + e_1, P_2 - e_2) = \ell_1$. We have that $\ell(P_2 - e_2, P_6 + e_1 - e_2) = \ell_2$ and that $\ell(P_4 + e_1, P_6 + e_1 - e_2) = \ell_3$. Then $[P_2 - e_2, P_4 + e_1, P_6 + e_1 - e_2] \equiv [P_1 - e_2, P_4 + e_1, P_2 - e_2]$.

13. Take the triangle $[P_1 - e_2, P_3, P_6 - e_2]$. We have $P_6 - e_2 = \left(-\frac{1}{7}, -\frac{5}{7}\right)$. We have already seen that $\ell(P_3, P_1 - e_2) = \ell_3$ and that $\ell(P_1 - e_2, P_6 - e_2) = \ell_2$. We have that $\ell(P_3, P_6 - e_2) = \ell_1$ and thus $[P_1 - e_2, P_3, P_6 - e_2] \equiv [P_2 - e_2, P_4 + e_1, P_6 + e_1 - e_2]$.

14. Take the triangle $[P_4, P_3, P_6 - e_2]$. We know that $\ell(P_3, P_6 - e_2) = \ell_1$ and that $\ell(P_4, P_3) = \ell_2$. Moreover, $\ell(P_4, P_6 - e_2) = \ell_3$, hence, $[P_4, P_3, P_6 - e_2] \equiv [P_1 - e_2, P_3, P_6 - e_2]$.

□

Remark 3.3.13. *The central configuration corresponds to the barycenter of $\Delta_x \times \Delta_y$. Indeed, the barycenter of Δ_x, bar_{Δ_x} is given by $bar_{\Delta_x} = \frac{1}{7}\sum_{i=1}^{7} v_i$ where v_i are the vertices of Δ_x. Computing, we find $bar_{\Delta_x} = \left(\frac{2}{7}, \frac{3}{7}, \frac{1}{7}, -\frac{2}{7}, -\frac{3}{7}, -\frac{1}{7}\right)$, corresponding to the first coordinate of points $P_1, \ldots, P_6$. On the other hand, $bar_{\Delta_y} = \frac{1}{7}\sum_{i=1}^{7} w_i$ where w_i are the vertices of Δ_y. We find hence $bar_{\Delta_y} = \left(\frac{3}{7}, \frac{1}{7}, -\frac{2}{7}, -\frac{3}{7}, -\frac{1}{7}, \frac{2}{7}\right)$, which corresponds to the second coordinates of points $P_1, \ldots, P_6$.*

3.4 Aligned configurations

Recall from chapter 1 that we have introduced by $\mathrm{L}_{\mathbb{T}^2}(M, \mathbb{E}^3)$ the space of linear embeddings of the Moebius torus M into $\mathbb{E}^3$. In chapter II, we have constructed a simplicial complex C isometric to $\mathbb{T}^2$ and a linear embedding of this complex into $\mathbb{E}^3$, showing that $\mathrm{L}_{\mathbb{T}^2}(C, \mathbb{E}^3)$ is non empty. We have conjectured that $L_{\mathbb{T}^2}(M, \mathbb{E}^3)$ is empty. Let $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. We denote by $L_{\mathcal{T}}(\mathbb{E}^3)$ the set of linear isometric embeddings of $\mathcal{T}$ in $\mathbb{E}^3$. We obviously have

$$L_{\mathbb{T}^2}(M, \mathbb{E}^3) = \bigcup_{\mathcal{T}} \mathrm{L}_{\mathcal{T}}(\mathbb{E}^3).$$

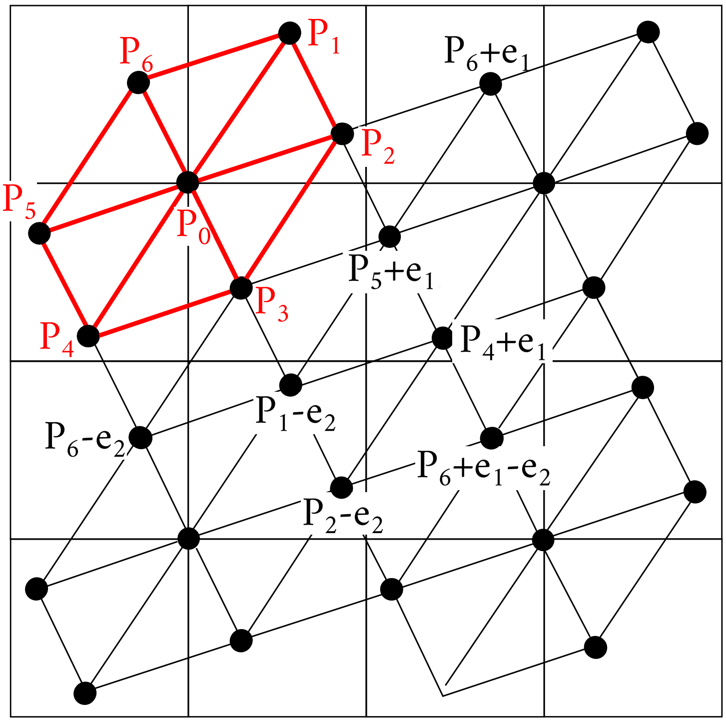

Figure 3.10: Lift of the central configuration

The aim of this section is to show that $L_{\mathcal{T}}(\mathbb{E}^3) = \emptyset$ for every aligned configuration $\mathcal{T}$ (see the definition 3.4.3).

Definition 3.4.1. *Let (jkl) and (jlm) be in LI, (the list of triangles in proposition 3.1.1) and let $[P_j, P_k, P_l]$ and $[P_j, P_l, P_m]$ be lifts of the corresponding triangles having a common edge $[P_j, P_l]$. We call* ***butterfly*** *the union of two such triangles, $[P_j, P_k, P_l] \cup [P_j, P_l, P_m]$ and we call their common edge $[P_j, P_l]$ the* ***body*** *of the butterfly (see figure 3.4). Using the terminology of simplicial complexes, a butterfly is the star of its body. We say that two butterflies are equivalent if and only if one of them is a translation of the other one by a translation of vector in $\mathbb{Z}e_1 + \mathbb{Z}e_2$.*

Notation: For short, we write $P_1[P_2, P_3]P_4$ for the butterfly $[P_1, P_2, P_3] \cup [P_2, P_3, P_4]$. We consider the 7 consecutive butterflies (see Figure 3.12):

- $B_0 := P_1[P_0, P_2]P_3$
- $B_1 := P_1^1[P_0^1, P_2^1]P_3^1 + e_1$
- $B_2 := P_1^2[P_0^2, P_2^2]P_3^2$

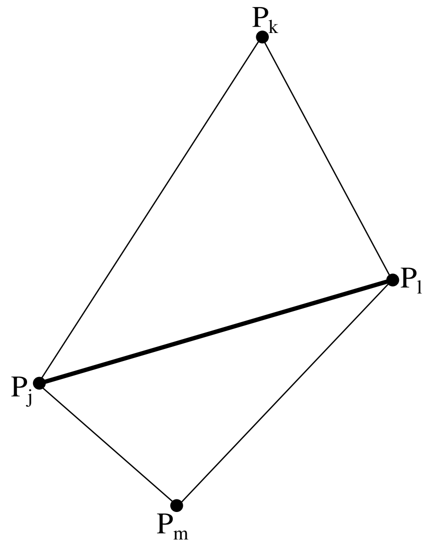

Figure 3.11: The butterfly $P_k[P_j, P_l]P_m$.

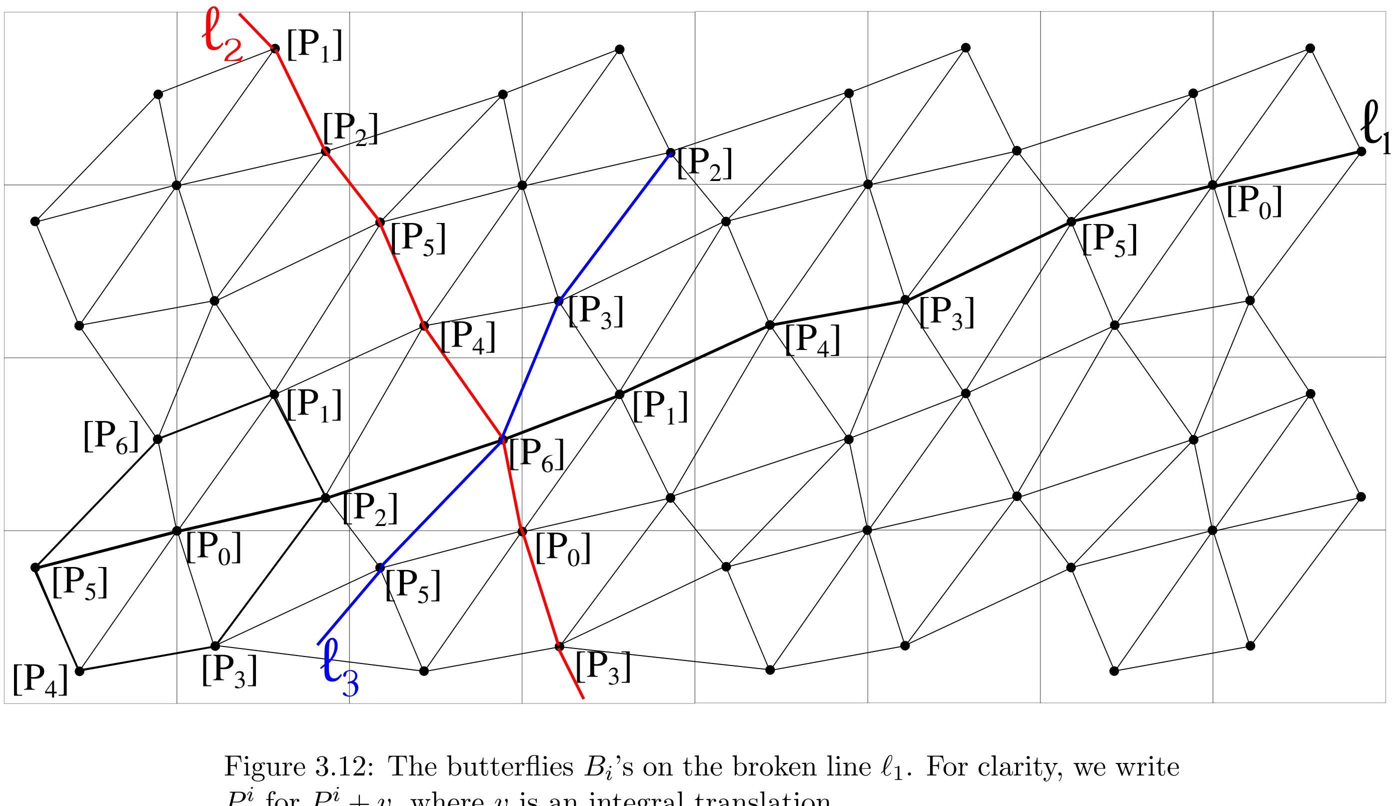

Figure 3.12: The butterflies B_i's on the broken line ℓ_1. For clarity, we write P_j^i for $P_j^i + v$, where v is an integral translation.

- $B_3 := P_1^3[P_0^3, P_2^3]P_3^3 + 2e_1 + e_2$
- $B_4 := P_1^4[P_0^4, P_2^4]P_3^4 + 2e_1 + e_2$
- $B_5 := P_1^5[P_0^5, P_2^5]P_3^5 + 3e_1 + e_2$
- $B_6 := P_1^6[P_0^6, P_2^6]P_3^6 + e_1,$

and we denote by $\mathcal{B} := \{B_i\}_{i=0}^6$. We denote by ℓ_1 the broken line formed by the bodies of the butterflies $B_0, \ldots, B_6$. They appear consecutively in the following order: $B_0, B_2, B_6, B_1, B_4, B_3$ and B_5.

Lemma 3.4.2. *All the butterflies of the set $\mathcal{B}$ are non equivalents. As a consequence, $\mathcal{B}$ contains a unique lift of $[p_j, p_k, p_l]$ for every $(jkl) \in LI$.*

Proof. The butterfly B_0 contains a lift of two triangles whose vertex indices are (012) and (023). The butterfly B_1 contains, by definition 3.3.6, a lift of the triangles $[p_1, p_5, p_4]$ and $[p_1, p_4, p_2]$. That is, triangles with indices (154) and (142). The butterfly B_2 contains a lift of $[p_2, p_4, p_6]$ and $[p_2, p_6, p_5]$, then triangles with indices (246) and (265). The butterfly B_3 contains lifts of triangles whose vertex indices are (325) and (351). The butterfly B_4, contains, by definition 3.3.6, lifts of triangles with indices (403) and (436). The butterfly B_5 contains two triangles whose indices correspond to (560) and (504). Finally, the lifts of the triangles contained in B_6 have indices (631) and (610). Note that the triangles in $\mathcal{B}$ have different indices two by two. Moreover, we only have one lift of a triangle with index (jkl) for every $(jkl) \in LI$ since LI has 14 elements and $\mathcal{B}$ contains 14 triangles. We deduce that no two triangles in $\mathcal{B}$ are equivalent. □

Let θ_i be the angle between the body of the butterfly B_i and the horizontal direction, for $i = 1, \ldots, 7$ (see figure 3.13).

Definition 3.4.3. *We say that an element $\mathcal{T}$ is an **aligned** configuration if for all $i = 0 \ldots, 6$, $\theta_i = \arctan \frac{1}{3}$.*

We denote by S the subspace of $C_{I,3} \subset \mathrm{GE}(M, \mathbb{T}^2)$ consisting of all aligned configurations (see figure 3.14). Observe that for every $\mathcal{T} \in S$, the bodies of the B_i's are aligned for $i = 0, \ldots, 6$. In particular, the *Central configuration* described in the previous section is in S. We have the following proposition for S:

Proposition 3.4.4. *S can be written as the graph of an affine map $f : \Delta_x \to \Delta_y$, where Δ_x and Δ_y are the simplices of theorem 3.3.9.*

The proof of this proposition is postponed at the end of this section. The main result of this section is the following:

Theorem 3.4.5. *There exists a 12-dimensional neighborhood $N(S)$ of S such that for every element $\mathcal{T}$ of $N(S)$, we have $L_{\mathcal{T}}(\mathbb{E}^3) = \emptyset$.*

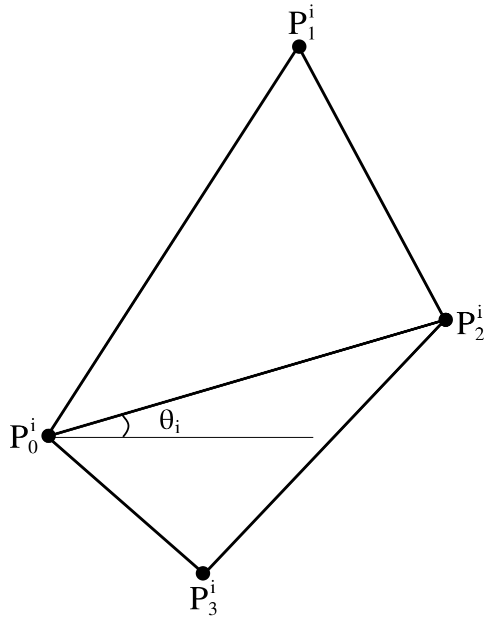

Figure 3.13: A butterfly and its angle.

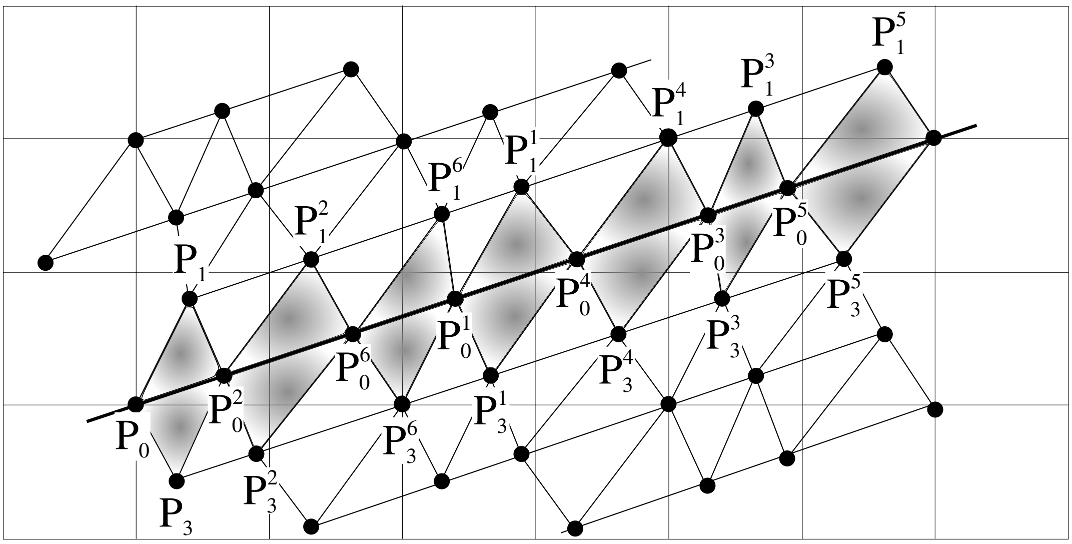

Figure 3.14: An element $\mathcal{T}$ of S. Here again, we write P_j^i for $P_j^i + v$, where v is an integral translation.

To prove this theorem we need some lemmas. We denote $AB := B - A$ for $A, B \in \mathbb{E}^n$, and we write $AB = (X_{AB}, Y_{AB})$. We will denote by $\|AB\|_n$, or simply $\|AB\|$, the norm of AB.

Definition 3.4.6. *We say that a butterfly $R_1[R_0, R_2]R_3$ of a configuration $\mathcal{T}$ is* ***realizable*** *in Euclidean space $\mathbb{E}^n$ if there exist four points $Q_0, Q_1, Q_2, Q_3 \in \mathbb{E}^n$ such that $d_{\mathbb{T}^2}(\pi(R_i), \pi(R_j)) = \|Q_iQ_j\|$ for every (i, j).*

Lemma 3.4.7. *Let $R_1[R_0, R_2]R_3$ be a butterfly of some configuration in $C_{I,3}$ (see figure 3.4). The butterfly is realizable in $\mathbb{E}^n$, for $n \geq 3$, if and only if*

$$\|R_1'R_3\|_2 \leq \|R_1''R_3\|_2,$$

where $R_1'' := R_1 - e_2$, and R_1' is the reflexion of R_1 through the segment $[R_0, R_2]$.

Proof. Consider four points $Q_0, Q_1, Q_2, Q_3 \in \mathbb{E}^n$ such that $d_{\mathbb{T}^2}(\pi(R_i), \pi(R_j)) = \|Q_iQ_j\|$ for every $(i, j) \neq (1, 3)$ with $i < j$. In particular, $\|Q_0Q_1\|^2 = \|R_0R_1\|^2$ and $\|Q_1Q_2\|^2 = \|R_1R_2\|^2$. Fixing Q_0, Q_2, Q_3, the extrema of the $\|Q_1Q_3\|^2$ seen as a function of Q_1 satisfy the Lagrange multiplier equation:

$$2\langle Q_1Q_3, .\rangle_{\mathbb{E}^n} = 2\lambda\langle Q_0Q_1, .\rangle_{\mathbb{E}^n} + 2\mu\langle Q_1Q_2, .\rangle_{\mathbb{E}^n}$$

for some λ, μ. Said differently, Q_1Q_3 is in the plane spanned by Q_0Q_1 and Q_1Q_2. The extrema of $\|Q_1Q_3\|$ are obtained when Q_0, Q_1, Q_2, Q_3 are coplanar. In Figure 3.4 this corresponds to the maximum $\|R_1R_3\|$ and the minimum $\|R_1'R_3\|$. Clearly, every intermediate distance can be realized as soon as $n \geq 3$ by pivoting triangle $R_0R_1R_2$ about the axis R_0R_2. Note that for any configuration in $C_{I,3}$ we have

$$d_{\mathbb{T}^2}(\pi(R_1), \pi(R_3)) = \|R_1''R_3\| < \|R_1R_3\|.$$

Hence, $R_1[R_0, R_2]R_3$ is realizable if and only if $\|R_1'R_3\| \leq \|R_1''R_3\|$. ☐

We denote by e_φ the unitary vector $(\cos\varphi, \sin\varphi)$, with $\varphi \in [0, 2\pi[$. We have the following:

Lemma 3.4.8. *The butterfly $P_1^i[P_0^i, P_2^i]P_3^i$ with angle θ_i is realizable if and only if*

$$4\cos^2\theta_i \left(Y_{P_1^iP_0^i} - X_{P_1^iP_0^i}\tan\theta_i\right)\left(Y_{P_3^iP_2^i} - X_{P_3^iP_2^i}\tan\theta_i\right) - 1 \leq 2Y_{P_1^iP_3^i}$$

Proof. By lemma 3.4.7, we have:

$$||P_1' - P_3^i||^2 \leq ||P_1^i - e_2 - P_3^i||^2 \tag{3.13}$$

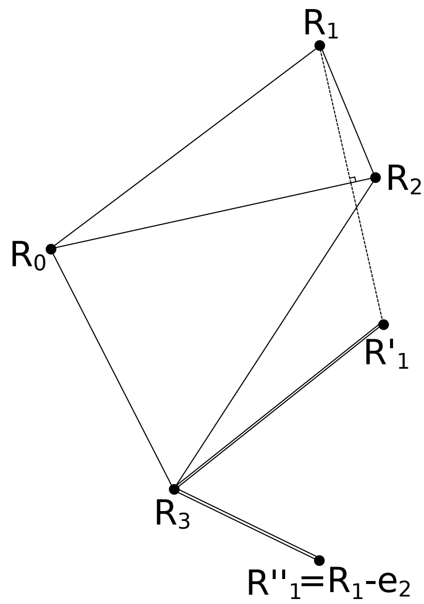

Figure 3.15: Butterfly with points R_1' and R_1'' as described above.

Now, the reflexion of P_1^i, P_1' through the segment $[P_0^i, P_2^i]$, can be written as $P_1' = P_1^i + 2\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle e_{\theta_i+\frac{\pi}{2}}$. Then, we have

$$\begin{aligned}
||P_1' - P_3^i||^2 &= ||P_1^i + 2\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle e_{\theta_i+\frac{\pi}{2}} - P_3^i||^2\\
&= ||P_1^i + 2\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle e_{\theta_i+\frac{\pi}{2}}||^2\\
&- 2\left\langle P_1^i + 2\left\langle P_1^iP_0^i, e_{\theta+\frac{\pi}{2}}\right\rangle e_{\theta_i+\frac{\pi}{2}}, P_3^i\right\rangle + ||P_3^i||^2\\
&= ||P_1^i||^2 + 4\left(\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\right)^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left\langle P_1^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\\
&- 2\left\langle P_1^i, P_3^i\right\rangle - 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left\langle P_3^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle + ||P_3^i||^2\\
&= ||P_1^i - P_3^i||^2 + 4\left(\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\right)^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left\langle P_1^i - P_3^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\\
&= ||P_1^iP_3^i||^2 + 4\left(\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\right)^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left\langle P_1^iP_3^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\\
&= ||P_1^iP_3^i||^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left(\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle - \left\langle P_1^iP_3^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\right)\\
&= ||P_1^iP_3^i||^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left(\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle + \left\langle P_3^iP_1^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\right)\\
&= ||P_1^iP_3^i||^2 + 4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle\left\langle P_3^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle
\end{aligned}$$

And we have also

$$\begin{aligned}||P_1^i - e_2 - P_3^i||^2 &= ||P_1^i - e_2||^2 - 2\left\langle P_1^i - e_2, P_3^i\right\rangle + ||P_3^i||^2 \\ &= ||P_1^i||^2 - 2\left\langle P_1^i, e_2\right\rangle + 1 - 2\left\langle P_1^i, P_3^i\right\rangle + 2\left\langle e_2, P_3^i\right\rangle + ||P_3^i||^2 \\ &= ||P_1^iP_3^i||^2 + 2\left\langle P_1^iP_3^i, e_2\right\rangle + 1\end{aligned}$$

We can write $P_3^iP_0^i = P_3^iP_2^i + P_2^iP_0^i$, then $\left\langle P_3^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle = \left\langle P_3^iP_2^i + P_2^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle = \left\langle P_3^iP_2^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle + \left\langle P_2^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle$, since $P_2^iP_0^i \perp e_{\theta_i+\frac{\pi}{2}}$, we have $\left\langle P_3^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle = \left\langle P_3^iP_2^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle$.

We have then that the condition of lemma 3.4.7 can be written as follows:

$$4\left\langle P_1^iP_0^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle \left\langle P_3^iP_2^i, e_{\theta_i+\frac{\pi}{2}}\right\rangle \leq 2\left\langle P_1^iP_3^i, e_2\right\rangle + 1 \tag{3.14}$$

Recall that $e_{\theta_i+\frac{\pi}{2}} = \left(\cos\left(\theta_i + \frac{\pi}{2}\right), \sin\left(\theta_i + \frac{\pi}{2}\right)\right) = (-\sin\theta_i, \cos\theta_i)$. We have the following inequality:

$$4\left(-X_{P_1^iP_0^i}\sin\theta_i + Y_{P_1^iP_0^i}\cos\theta_i\right)\left(-X_{P_3^iP_2^i}\sin\theta_i + Y_{P_3^iP_2^i}\cos\theta_i\right) - 1 \leq 2Y_{P_1^iP_3^i}$$

or equivalently:

$$4\cos^2\theta_i\left(-X_{P_1^iP_0^i}\tan\theta_i + Y_{P_1^iP_0^i}\right)\left(-X_{P_3^iP_2^i}\tan\theta_i + Y_{P_3^iP_2^i}\right) - 1 \leq 2Y_{P_1^iP_3^i} \tag{3.15}$$

□

We can now prove Theorem 3.4.5:

Proof of theorem 3.4.5. We first consider the case where the considered configuration is in S. Since the angle of each butterfly in $\mathcal{B}$ is $\arctan\frac{1}{3}$, we have that for every i the following points

$$P_0^i, P_2^i, P_6^i + e_1, P_1^i + e_1, P_4^i + 2e_1 + e_2, P_3^i + 2e_1 + e_2, P_5^i + 3e_1 + e_2$$

lie on a line with slope $\frac{1}{3}$. Suppose that each butterfly B_i, or equivalently $P_1^i[P_0^i, P_2^i]P_3^i$, is realizable. Then (3.15) is satisfied. Using that $\cos^2\theta_i = \frac{9}{10}$ we may thus write

$$\frac{18}{5}\left(Y_{P_1^iP_0^i} - \frac{X_{P_1^iP_0^i}}{3}\right)\left(Y_{P_3^iP_2^i} - \frac{X_{P_3^iP_2^i}}{3}\right) - 1 \leq 2Y_{P_1^iP_3^i}$$

From the above points alignments, we deduce that the vectors $P_1^iP_0^i - e_1$ and $P_3^iP_2^i - 2e_1 - e_2$ have slope $\frac{1}{3}$. It follows that

$$Y_{P_1^iP_0^i} = \frac{X_{P_1^iP_0^i} - 1}{3} \quad \text{and} \quad Y_{P_3^iP_2^i} - 1 = \frac{X_{P_3^iP_2^i} - 2}{3}$$

whence

$$Y_{P_1^iP_0^i} - \frac{X_{P_1^iP_0^i}}{3} = -\frac{1}{3} \quad \text{and} \quad Y_{P_3^iP_2^i} - \frac{X_{P_3^iP_2^i}}{3} = \frac{1}{3}$$

Replacing in (3.15), we get

$$-\frac{7}{10} \leq Y_{P_1^iP_3^i} = Y_{P_3^i} - Y_{P_1^i} \tag{3.16}$$

With the help of the dictionary 3.3.6, we compute

$$\begin{aligned}
Y_{P_1^0P_3^0} &= Y_{P_3} - Y_{P_1}\\
Y_{P_1^1P_3^1} &= Y_{P_2} - Y_{P_5} - 1\\
Y_{P_1^2P_3^2} &= Y_{P_5} - Y_{P_4} - 1\\
Y_{P_1^3P_3^3} &= Y_{P_1} - Y_{P_2} - 1\\
Y_{P_1^4P_3^4} &= Y_{P_6} - Y_{P_0} - 1\\
Y_{P_1^5P_3^5} &= Y_{P_4} - Y_{P_6}\\
Y_{P_1^6P_3^6} &= Y_{P_0} - Y_{P_3} - 1
\end{aligned}$$

Summing inequation (3.16) over all i's we obtain

$$-\frac{7}{10} \times 7 \leq -5,$$

or equivalently $49 \geq 50$, which is an obvious contradiction. We conclude that at least one of B_i is not realizable, so that every configuration in S is not realizable in $\mathbb{E}^n$, for $n \geq 3$. More precisely, we have proved that for all configurations in S

$$\sum_{i=0}^{7} \left(4\cos^2\theta_i \left(-X_{P_1^iP_0^i}\tan\theta_i + Y_{P_1^iP_0^i}\right)\left(-X_{P_3^iP_2^i}\tan\theta_i + Y_{P_3^iP_2^i}\right) - 1 - 2Y_{P_1^iP_3^i}\right) = \frac{1}{10}$$

By continuity, the left hand side must be strictly positive in some neighborhood $N(S)$ of S. This implies that (3.15) is not satisfied for at least one i, i.e. that B_i is not realizable. We conclude that $L_{\mathcal{T}}(\mathbb{E}^3) = \emptyset$ for every $\mathcal{T} \in N(S)$. □

We prove the proposition 3.4.4:

Proof of proposition 3.4.4. To simplify the notations, we introduce the following points.

- $R_0 := P_0$
- $R_1 := P_2$
- $R_2 := P_6 + e_1$
- $R_3 := P_1 + e_1$
- $R_4 := P_4 + 2e_1 + e_2$
- $R_5 := P_3 + 2e_1 + e_2$
- $R_6 := P_5 + 3e_1 + e_2$

Note that the set $\{R_i\}_{i=0}^6$ contains one and only one lift of $p_i \in \mathbb{T}^2$ for every $i = 0, \ldots, 6$. We denote $R_i = (X_i, Y_i)$. From 3.3.2, it is easily seen that a configuration is in $C_{I,3}$ if and only if

I. $X_1 - X_0 < \frac{1}{2}$

II. $X_2 - X_1 < \frac{1}{2}$

III. $X_3 - X_1 < \frac{1}{2}$

IV. $X_4 - X_3 < \frac{1}{2}$

V. $X_5 - X_4 < \frac{1}{2}$

VI. $X_6 - X_0 < \frac{1}{2}$

VII. $X_0 - X_6 < -\frac{5}{2}$

I'. $Y_3 - Y_0 < \frac{1}{2}$

II'. $Y_6 - Y_3 < \frac{1}{2}$

III'. $Y_2 - Y_6 < -\frac{1}{2}$

IV'. $Y_5 - Y_2 < \frac{1}{2}$

V'. $Y_1 - Y_5 < -\frac{1}{2}$

VI'. $Y_4 - Y_1 < \frac{1}{2}$

VII'. $Y_0 - Y_4 < -\frac{1}{2}$

Such a configuration is in S if and only if it moreover satisfies

$$Y_i = \frac{X_i}{3} \tag{3.17}$$

for $i = 0, \ldots, 6$. These last equations express the alignment of the R_i's. A configuration in S is thus a point of the graph of the map $f : \Delta_x \to \mathbb{R}^6$ given by $f(X_1, \ldots, X_6) = \left(\frac{X_1}{3}, \ldots, \frac{X_6}{3}\right)$. Here we have used the coordinates $(R_1, \ldots, R_6)$ instead of $(P_1, \ldots, P_6)$ for a configuration. Conversely, consider a configuration $(R_1, \ldots, R_6) = ((X_1, \ldots, X_6), f(X_1, \ldots, X_6))$ in the graph of f. In order to show that this configuration is in S it is enough to prove that $f(X_1, \ldots, X_6) \in \Delta_y$. Now, we have that $Y_i = \frac{X_i}{3}$, and the equations $I - VI$ imply that

$$Y_{i+1} - Y_i < \frac{1}{6}$$

for $i = 0, \ldots, 5$, while equation VII implies

$$Y_0 - Y_6 < -\frac{5}{6}.$$

. We now verify that these inequalities imply equations $I' - VII'$:

- $Y_3 - Y_0 = (Y_3 - Y_2) + (Y_2 - Y_1) + (Y_1 - Y_0) < 3 \times \frac{1}{6} = \frac{1}{2}$.
- $Y_6 - Y_3 = (Y_6 - Y_5) + (Y_5 - Y_4) + (Y_4 - Y_3) < 3 \times \frac{1}{6} = \frac{1}{2}$

- $Y_2 - Y_6 = -(Y_6 - Y_5) - (Y_5 - Y_4) - (Y_4 - Y_3) - (Y_3 - Y_2) < -4 \times \frac{1}{6} = -\frac{2}{3} < -\frac{1}{2}$
- $Y_5 - Y_2 = (Y_5 - Y_4) + (Y_4 - Y_3) + (Y_3 - Y_2) < 3 \times \frac{1}{2} = \frac{1}{2}$
- $Y_1 - Y_5 = -(Y_5 - Y_4) - (Y_4 - Y_3) - (Y_3 - Y_2) - (Y_2 - Y_1) < -4 \times \frac{1}{6} < -\frac{1}{2}$
- $Y_4 - Y_1 = (Y_4 - Y - 3) + (Y_3 - Y_2) = (Y_2 - Y_1) < 3 \times \frac{1}{6} = \frac{1}{2}$
- $Y_0 - Y_4 = -(Y_4 - Y_3) - (Y_3 - Y_2) - (Y_2 - Y_1) - (Y_1 - Y_0) = -4 \times \frac{1}{6} < -\frac{1}{2}$

This proves that $f(\Delta_x) \subset \Delta_y$. □

Chapter 4

Numerical experiments

Through this work, several computational programs have been used to fix ideas about the realizability of a random triangulation $\mathcal{T}$ in $C_{I,3}$. In this chapter we give the results obtained from the numerical exploration as well as the mathematical method used. All the programs were written in Python. To write the programs we use some results of the Gramian matrix that we now recall.

4.1 Method used

Let $v_1, \dots, v_k$ a set of k vectors in $\mathbb{E}^n$, their **Gramian matrix** is defined to be:

$$G(v_1, \dots, v_k) := \left(\langle v_i, v_j \rangle\right)_{ij}.$$

The **Gramian** of $v_1, \dots, v_k$ is the determinant of $G(v_1, \dots, v_k)$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{E}^n$ and A the matrix of $v_1, \dots, v_k$ in this basis, i.e., $A = (v_1, \dots, v_k)$. We denote by A^T the transpose of A, then we have

$$G(v_1, \dots, v_k) = A^T A.$$

Since $\det(A^T A) = \det(A)^2$, we deduce that $\det G(v_1, \dots, v_k) \geq 0$.

If $k = n$, since the n-volume of the parallelotope spanned by $v_1, \dots, v_n$ is given by $\mathrm{vol}(v_1, \dots, v_n) = |\det(A)|$, we have that

$$\det G(v_1, \dots, v_n) = \mathrm{vol}^2(v_1, \dots, v_n)$$

.

In particular, if $v_1, \dots, v_n$ are linearly dependent, then $\det(v_1, \dots, v_n) = 0$. In general, we have the following property (see [Bar99]):

Proposition 4.1.1. *Let $v_1, \dots, v_k \in \mathbb{E}^n$, we have*

$$det\, G(v_1, \dots, v_k) = vol_k^2(v_1, \dots, v_k)$$

where $vol_k(v_1, \ldots, v_k)$ *is the* k*-dimensional volume of the parallelotope generated by* $v_1, \ldots, v_k$*. In particular, if* $k > n$ *then* $\det G(v_1, \ldots, v_k) = 0$*.*

Let $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$. We denote by ℓ_{ij} the distance $d_{\mathbb{T}^2}(P_i, P_j)$ and by ℓ_{0i} the distance $d_{\mathbb{T}^2}(P_0, P_i)$. If in addition there is a linear embedding f of $\mathcal{T}$ into $\mathbb{E}^3$, i t defines 6 vectors in $\mathbb{E}^n$:

$$v_i = f(P_0)f(P_i)$$

and by the preservation of distances of the lengths we have

$$||v_i|| = d(P_0, P_i) = \ell_{0i}$$

$$||v_j - v_i|| = d(P_i, P_j) = \ell_{ij}$$

.

We now introduce a new $k \times k$ matrix $\Gamma(\ell_{ij})$defined by.

$$\Gamma(\ell_{ij}) := \frac{1}{2}\left(\ell_{0i}^2 + \ell_{0j}^2 - \ell_{ij}^2\right)_{ij}.$$

Observe that this matrix only depends on the triangulation $\mathcal{T}$. Since we have assumed that $\mathcal{T}$ admits an linear isometric embedding in $\mathbb{E}^n$ we must have:

$$G(v_1, \ldots, v_6) = \Gamma(\ell_{ij}) \tag{4.1}$$

since $\langle v_i, v_j \rangle = \frac{1}{2}\left(\ell_{0i}^2 + \ell_{0j}^2 - \ell_{ij}^2\right)$.

The equality 4.1 could be used to prove that a given $\mathcal{T} \in \mathrm{GE}(M, \mathbb{T}^2)$ admits a linear isometric embedding in $\mathbb{E}^n$. Indeed, if such an embedding exists, then

$$\det G(v_{i_1}, \ldots, v_{i_k}) > 0$$

for every $1 \leq k \leq n$ and every k-subset $I_k = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, 6\}$. In particular, if some minor of $\Gamma(\ell_{ij})$ satisfies

$$\det\left(\Gamma(\ell_{ij})_{i,j \in I_k}\right) < 0,$$

then $\mathcal{T}$ admits no linear embedding.

4.2 Computing Gramians

The first program entitled **toremoebius.py** explores the Gramians of 7 vertices $P_0, \ldots, P_6$ (with $P_0 = (0, 0)$) of a random configuration in $C_{I,3}$.

Let $\mathcal{P} \in \mathbb{R}^{12}$ be defined by the successive coordinates of $P_1, \ldots, P_6$. The program takes a random admissible point $\mathcal{P} \in \mathbb{R}^{12}$ and computes the Gramian associated to it. It also computes the signs of the minors of size 4 and 5 of $\Gamma(\ell_{ij})$. We have run several tests and observed in every case that at least one minor of size 4 was negative. In fact, we observed that one of these negative minors corresponded to one of the 7 butterflies with body on ℓ_1. This is the case for the central configurations where the negative minors of size 4 are equal and correspond precisely to the 7 butterflies with body on ℓ_1. We also checked with the program **rationalpoint.py** that for all the 49 vertices of the configuration space $C_{I,3}$ one of the seven butterflies is non realizable.

These observations motivated to explore more systematically the 7 minors of size 4 associated to the butterflies $B_0, \ldots, B_6$ along ℓ_1. The program entitled **pointshazard.py** calculates amongst a number of n configurations, how many of them have 0 out of the 7 butterflies with negative determinant, how many of these configurations have only one butterfly with negative determinant and so on until 7 butterflies. We have run this program with n up to 10^7 and collected the results in the following table. Row i indicates how many of the 10^7 random configurations have i negative minors among the 7.

Number of butterflies with negative Gramian	Number of configurations
0	0
1	10
2	5725
3	251718
4	2484910
5	4919382
6	2124913
7	213342

These results consolidate our conjecture that at least one the 7 butterflies is not realizable. We should remark, however, that the number of experiments, is very small with respect to the dimension of the the space of configurations which is 12-dimensional. Indeed, sampling as few as 4 points in every dimension already gives $4^{12} \simeq 1.6\,10^7$ configurations to be tested.

4.3 Exploring geometric and algebraic characteristics of 7 butterflies

We first tried to identify thanks to some geometrical characteristics one among the 7 butterflies that would always be non realizable. For this, we first recall the criterion for realizability of Lemma 3.4.7.

Let $A[B,C]D$ a butterfly as seen in figure 4.1, A' the reflexion of A through the segment $[B,C]$ and A'' the point $A - e_2$. Then, $A[B,D]C$ is

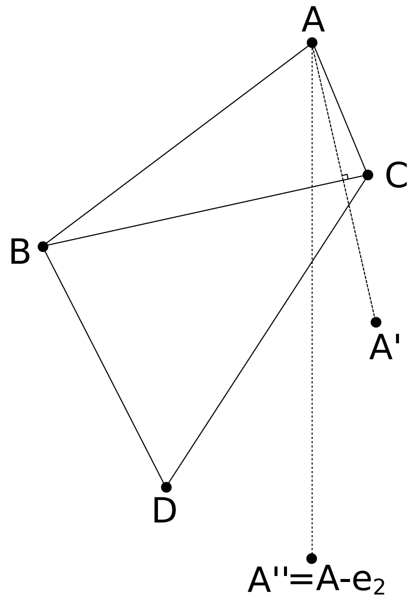

Figure 4.1: Butterfly $A[B, C]D$.

realizable if and only if $||A'D|| \leq ||A''D||$. It is plausible that the butterfly for which $||A'D||$ is minimal always fails the test, hence is not realizable. Unfortunately, we wrote a program **calcul-distances-arets.py** and we found examples that contradict this intuition. We tried other criteria in the same spirit, but none of them have succeeded.

We have also computed the sum of the seven Gramians associated to the butterflies hoping that this sum would be negative. This would indeed have implied that at least one of the butterflies is not realizable. The program **det-sum.py** computes this sum. However, the result of this experiment was again negative, as we found a positive sum for some admissible configurations.

We already saw that the central configuration studied in section 3 of chapter 2 is non realizable. We know even more, by lemma 3.3.12 we have that all butterflies are congruent, therefore every butterfly in the central configuration is non realizable. We also saw that the 49 vertices of the configuration space $C_{I,3}$ have a non realizable butterfly. One approach to extend this to all configurations is to prove (1) the non realizability on the boundary of $C_{I,3}$ and (2) that if a boundary configuration c has a non realizable butterfly, this butterfly remains unrealizable on the whole segment S_c that connects c to the central configuration. We were unable to show any of these two points. To show the difficulty of point (2) we have plotted the values of the 7 Gramians along the segment S_c where c has only one negative

Gramian. See Figure 4.2.

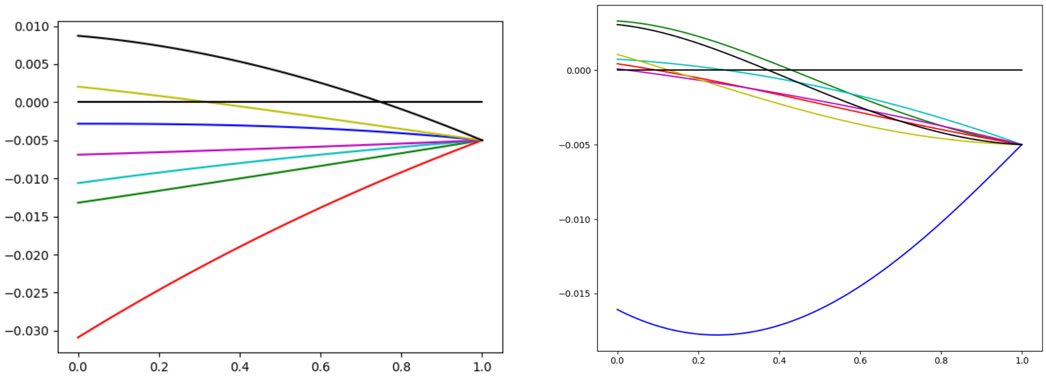

Figure 4.2: Plot of the 7 Gramians along the segment S_c. The abscissa is the interpolation parameter between 0 (for c) to 1 (for the central configuration). The vertical axis represents the value of the Gramian. We also display the constant zero value for reference. Left, a typical case for c where 5 out of 7 Gramians are negative. Right, all the Gramians of c except 1 are positive! Note that the curve for the negative Gramian is not monotone.

4.4 Visualization

In order to facilitate the exploration of the configuration space we wrote some visualization tools. We can first visualize any configuration by plotting its seven vertices in D_0 as on Figure 4.3. We also wrote a program to draw part of the lift of the triangulation corresponding to a configuration. See Figure 4.4.

From the butterfly in Figure 4.1, we know that $A[B,C]D$ is realizable if and only if $||A'D|| \leq ||A''D||$. The same criteria is true if we change the role of points D and A, that is, $A[B,C]D$ is realizable if and only if $||D'A|| \leq ||D''A||$, where D' is the reflexion of D through the segment $[B,C]$ and $D'' = D + e_2$. We consider the bisector line between D' and D''. Obviously, $||D'A|| \leq ||D''A||$ if and only if A is on the same side as D' with respect to this line (see figure FIGURE). We can thus visualize pictorially if a butterfly is realizable by drawing this bisector line. The program **pavagepapillon.ggb** written in geoGebra shows a configuration with the bisector lines of the seven butterflies. The user can move each of the points $P_1, P_2, \ldots, P_6$ to generates different configurations (see figure 4.6) and see in realtime if the configuration is realizable.

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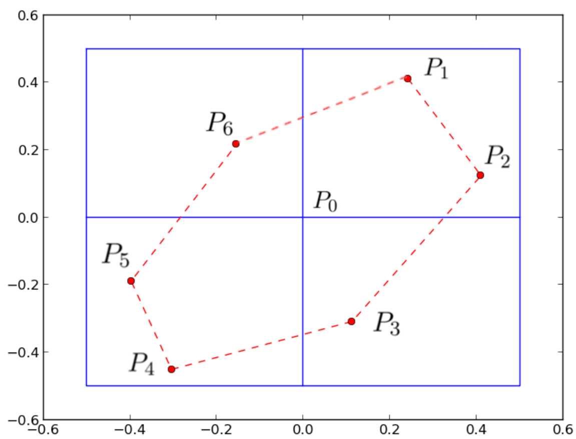

Figure 4.3: Plot of the seven points $P_0, \ldots, P_6$ in D_0 of an admissible configuration.

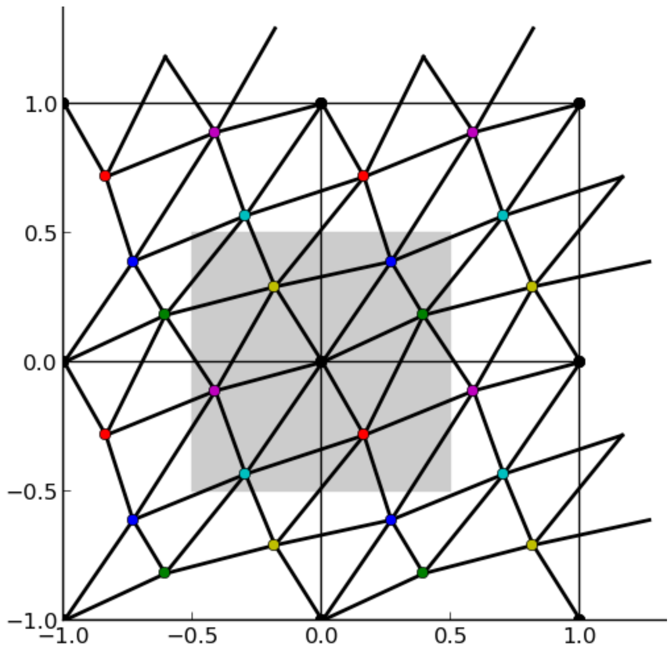

Figure 4.4: Part of the triangulation of the plane defined by the lift of a configuration.

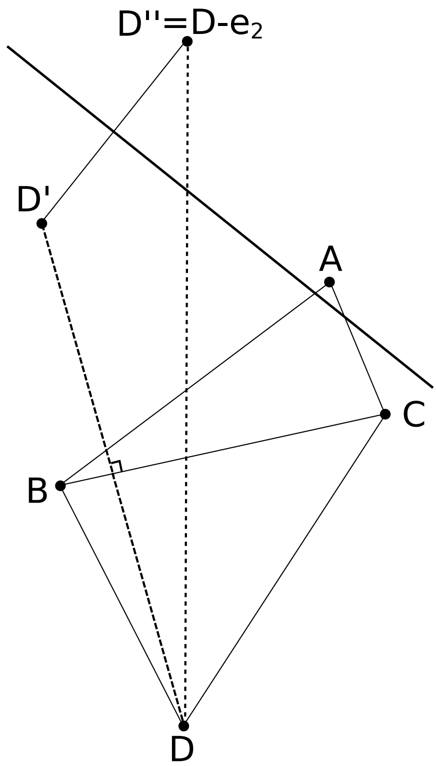

Figure 4.5: A butterfly with and the bisector of D' and D''.

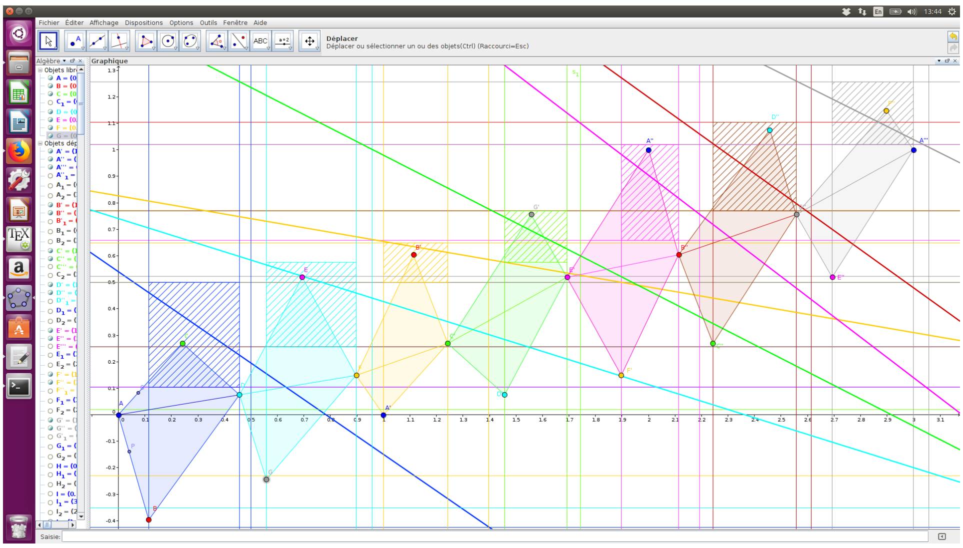

Figure 4.6: An admissible configuration showing which butterflies are non realizable.

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Plongements polyédriques du tore carré plat

Résumé: Dans cette thèse, on a construit un plongement isometrique PL du tore dans $\mathbb{E}^3$ et on a s'est interesé au plongements du tore de Moebius dans $\mathbb{E}^n$. On a montré que dans un certain espace de dimension 12, il n'y a aucune triangulation du tore de Moebius qui puisse être plongée dans $\mathbb{E}^n$.

Mots clés: Plongement; Tore de Moebius; Tore carré plat.

Polyhedral embeddings of the Flat Square Torus

Abstract: In this work we constructed a PL isometric embedding of the flat square torus into $\mathbb{E}^3$. Moreover, we showed that for a 12-dimensional space, there are no triangulation of the Moebius Torus that can be embedded into $\mathbb{E}^n$.

Keywords: Embedding; Moebius Torus; Flat Square Torus.

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