Flat tori in three dimensional space and convex integration

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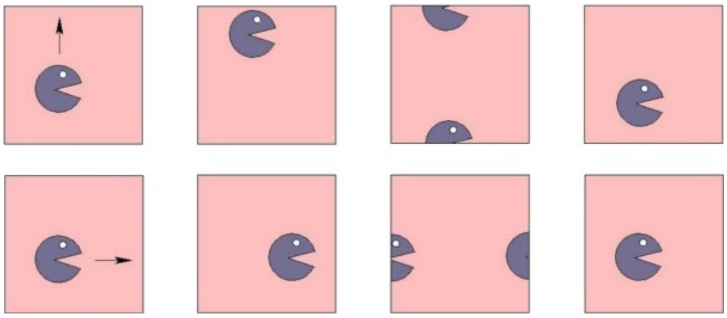
A Flat Torus in Three Dimensional Space!

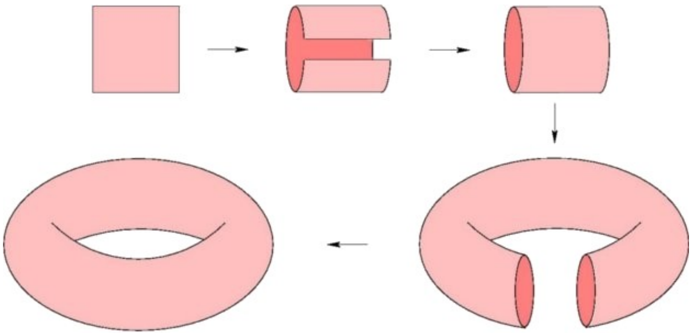
The facts. A team of four mathematicians and computer scientists from the Camille Jordan Institut (CNRS, Lyon I, INSA Lyon and Centrale Lyon), the GIPSA-lab (CNRS, Grenoble 1, Grenoble INP, Grenoble 3) and the Jean Kuntzmann laboratory (CNRS, Grenoble 1, Grenoble 2, Grenoble INP et INRIA) has produced the first images of a flat torus in three dimensional space. The images reveal an unexpected object, halfway between fractals and ordinary surfaces: a smooth fractal.

What is a flat torus? This is a square whose sides are pairwise identified. As a consequence, an imaginary being living in this square would exit the upper side each time it enters the lower side and similarly for the left and right sides.

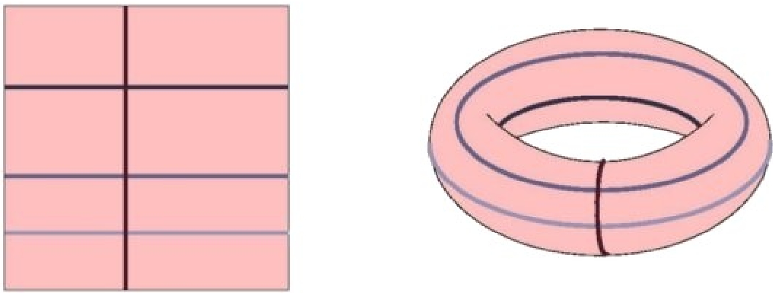


This imaginary square evokes some computer games where the characters desappear on one side of the screen and reappear on the opposite side. In Mathematics, this world is called the *square flat torus;* this is a particular flat torus.

What does it mean to visualize a flat torus in three dimensional space? In order to get a concrete idea of the square world, let us imagine that the square is made of some soft deformable material. We can now bend the square in the third dimension so as the upper and lower sides coincide. We obtain this way a cylinder. We can further bend the cylinder until its two boundaries are made face to face and can be stitched together. The resulting object looks like a bicycle inner tube: the square world, although abstract, is nothing more than a ring buoy!



This description of a square flat torus might be enlightening but suffers a major defect: It does not respect distances. Small distances in the square world may become large on the buoy.



To visualize the square flat torus is to find a representation of this torus that preserves distances.

Why is it important? It was a long-held belief that there could not exist any way to represent the square flat torus without distorting distances. It was thus a surprise for the specialists when, in the mid fifties, N. Kuiper and the Nobel prize J. Nash succeeded to prove that such a representation indeed exists and has fascinating and paradoxical properties. Visualizing a square flat torus allowed us to discover those enigmatic representations, to better understand their strange structure and to bring to light a new class of objects with unsuspected geometry. Beyond the geometry questions, the effective realization of a square flat torus supplied evidence that some familly of mathematical equations underlying the considered problem, could be solved with a computer. This fact was a priori far from evident.

Why, in the fifties, did Nash and Kuiper works not permit to visualize a flat square? Nash and Kuiper proved the *existence* of a representation that does not perturb the lenghts in the square flat torus. For a long time, this existence remained a challenge for the imagination of mathematicians. But *proving* and *showing* should sometimes be clearly distinguished in Mathematics. This is well explained by the thief allegory: Let us assume that a group of people is gathered around a jewel in a closed room. Let us further suppose that the light is turned off for a moment and that the jewel has disappeared when the light is again turned on. We then have the *proof* that a robber is hiding among the attendance but he can not be *exhibited*. Although the proofs of Nash and Kuiper are much more than an « existential » trap, their proofs do not provide a sufficiently explicit procedure that would allow for visualization or simply for a mental picture of a square flat torus.

What methods were used to visualize the square flat torus? After the fifties, the Abel prize M. Gromov shall change the course of the theory. In the 70-80's this mathematician, revisiting the results of Nash and Kuiper, extracted the underlying method of their works to generalize and enlighten their procedure. This method, called Convex Integration, also turned out to be remarkably efficient for other problems in Geometry. Convex Integration has built a reputation of abstraction that may not be disproportionate, although it hides the essential: convex integration does not only yield the existence of a solution, it also provides us with an effective construction. We have adapted this technique to obtain an algorithm for the representation of the square flat torus in three dimensional space. The implementation led us to the first images of a square flat torus. To our knowledge this is the first time convex integration is actually implemented.

What was discovered when analyzing the first images of the square flat torus? Mathematicians were puzzled by the works of Nash and Kuiper. These works could indeed prove the existence of objects whose regularity was problematic, if not paradoxical. They had to be smooth and rough at the same time... In effect, the mathematical analysis of the images reveals a surface belonging to two antagonist worlds; the smooth surfaces and the fractals, infinitely broken. When zooming in, we invariably observe ripples at smaller and smaller scales. Each ripple - called a *corrugation* – appears smooth when viewed alone, but the accumulation of those creates an object with a rough and fractal aspect.

What **difficulties were being experienced?** On the mathematical side, the effort was essentially focused on the simplification of the convex integration technique and on the analysis of the geometric structure of the produced images. Those efforts allowed us to bring into light a class of objects whose geometry lies inbetween fractals and smooth surfaces. On the computing side, one of the main difficulties was to deal with a very large amount of data. Our images of the flat square torus have required a grid mesh with two milliards nodes. The pictures were computed thanks to CIMENT, a center for computations in Grenoble.

Why a multidisciplinary team for a math project? The visualization of a square flat torus is the result of a six year old collaborative work: the HEVEA Project. It involves researchers from three different specialties: pure mathematics, applied mathematics and computer science. If the problem of representing a flat torus in 3D is a question of pure mathematics, the algorithmic part of the construction belongs to computer science while numerical issues in the implementation resort to applied mathematics. The project would not have succeeded without those different skills.

What are the perspectives opened up by this work? Convex integration has applications way beyond visualizing flat tori. It is used has a theoretical tool to determine atypical solutions of partial differential equations. Demonstrating that convex integration can be implemented open new perspectives in applied mathematics, notably for solving differential systems originating from Physics and Biology.

More specifically, our images reveal a class of objects whose structure lies inbetween smooth surfaces and fractals. Such objects could play a central rôle for shape analysis. They could also resolve some unexplained paradoxes.

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